Damped Free Vibrations
Single Degree of Freedom Systems

Introduction:
Damping – dissipation of energy.
For a system to vibrate, it requires energy. During vibration of the system, there will be continuous transformation of energy. Energy will be transformed from potential/strain to kinetic and vice versa.
In case of undamped vibrations, there will not be any dissipation of energy and the system vibrates at constant amplitude. Ie, once excited, the system vibrates at constant amplitude for infinite period of time. But this is a purely hypothetical case. But in an actual vibrating system, energy gets dissipated from the system in different forms and hence the amplitude of vibration gradually dies down. Fig.1 shows typical response curves of undamped and damped free vibrations.

Types of damping:
(i) Viscous damping
In this type of damping, the damping resistance is proportional to the relative velocity between the vibrating system and the surroundings. For this type of damping, the differential equation of the system becomes linear and hence the analysis becomes easier. A schematic representation of viscous damper is shown in Fig.2.

Here, $F \propto \dot{x}$ or $F = c\dot{x}$, where, $F$ is damping resistance, $\dot{x}$ is relative velocity and $c$ is the damping coefficient.
(ii) Dry friction or Coulomb damping
In this type of damping, the damping resistance is independent of rubbing velocity and is practically constant.

(iii) Structural damping
This type of damping is due to the internal friction within the structure of the material, when it is deformed.

Spring-mass-damper system:

Fig.3 shows the schematic of a simple spring-mass-damper system, where, m is the mass of the system, k is the stiffness of the system and c is the damping coefficient.
If x is the displacement of the system, from Newton’s second law of motion, it can be written

\[ m\ddot{x} = -c\dot{x} - kx \]
I.e \( m\ddot{x} + c\dot{x} + kx = 0 \) \hspace{1cm} (1)

This is a linear differential equation of the second order and its solution can be written as

\[ x = e^{st} \] \hspace{1cm} (2)

Differentiating (2),

\[ \frac{dx}{dt} = \dot{x} = se^{st} \]

\[ \frac{d^2x}{dt^2} = \ddot{x} = s^2 e^{st} \]

Substituting in (1),

\[ ms^2 e^{st} + cse^{st} + ke^{st} = 0 \]
\[ (ms^2 + cs + k)e^{st} = 0 \]

Or \[ ms^2 + cs + k = 0 \] \hspace{1cm} (3)

Equation (3) is called the characteristic equation of the system, which is quadratic in s. The two values of s are given by

\[ s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \] \hspace{1cm} (4)
The general solution for (1) may be written as
\[ x = C_1 e^{s_1 t} + C_2 e^{s_2 t} \] (5)
Where, \( C_1 \) and \( C_2 \) are arbitrary constants, which can be determined from the initial conditions.

In equation (4), the values of \( s_1 = s_2 \), when \( \left( \frac{c}{2m} \right)^2 = \frac{k}{m} \)

Or,
\[ \left( \frac{c}{2m} \right)^2 = \sqrt{\frac{k}{m}} = \omega_n \] (6)

Or \( c = 2m\omega_n \), which is the property of the system and is called critical damping coefficient and is represented by \( c_c \).

Ie, critical damping coefficient = \( c_c = 2m\omega_n \)
The ratio of actual damping coefficient \( c \) and critical damping coefficient \( c_c \) is called damping factor or damping ratio and is represented by \( \zeta \).

Ie, \( \zeta = \frac{c}{c_c} \) (7)

In equation (4), \( \frac{c}{2m} \) can be written as \( \frac{c}{2m} = \frac{c}{c_c} \times \frac{c_c}{2m} = \zeta \omega_n \)

Therefore, \( s_{1,2} = -\zeta \omega_n \pm \sqrt{(\zeta \omega_n)^2 - \omega_n^2} = \left[ -\zeta \pm \sqrt{\zeta^2 - 1} \right] \omega_n \) (8)

The system can be analyzed for three conditions.

(i) \( \zeta > 1 \), ie, \( c > c_c \), which is called over damped system.
(ii) \( \zeta = 1 \), ie, \( c = c_c \), which is called critically damped system.
(iii) \( \zeta < 1 \), ie, \( c < c_c \), which is called under damped system.

Depending upon the value of \( \zeta \), value of \( s \) in equation (8), will be real and unequal, real and equal and complex conjugate respectively.

(i) Analysis of over-damped system (\( \zeta > 1 \)).

In this case, values of \( s \) are real and are given by
\[ s_1 = \left[ -\zeta + \sqrt{\zeta^2 - 1} \right] \omega_n \] and \[ s_2 = \left[ -\zeta - \sqrt{\zeta^2 - 1} \right] \omega_n \]

Then, the solution of the differential equation becomes
\[ x = C_1 e^{\left[ -\zeta + \sqrt{\zeta^2 - 1} \right] \omega_n t} + C_2 e^{\left[ -\zeta - \sqrt{\zeta^2 - 1} \right] \omega_n t} \] (9)

This is the final solution for an over damped system and the constants \( C_1 \) and \( C_2 \) are obtained by applying initial conditions. Typical response curve of an over damped system is shown in fig.4. The amplitude decreases exponentially with time and becomes zero at \( t = \infty \).
(ii) Analysis of critically damped system ($\zeta = 1$).

In this case, based on equation (8), $s_1 = s_2 = -\omega_n$

The solution of the differential equation becomes

\begin{align*}
  x &= C_1e^{\xi t} + C_2te^{\xi t} \\
  I.e., \quad x &= C_1e^{-\omega t} + C_2te^{-\omega t} \\
  \text{Or,} \quad x &= (C_1 + C_2t)e^{-\omega t} \quad (10)
\end{align*}

This is the final solution for the critically damped system and the constants $C_1$ and $C_2$ are obtained by applying initial conditions. Typical response curve of the critically damped system is shown in fig.5. In this case, the amplitude decreases at much faster rate compared to over damped system.

![Graph showing displacement-time plots of over-damped and critically damped systems with zero starting velocity]
(iii) Analysis of under damped system ($\zeta < 1$).

In this case, the roots are complex conjugates and are given by

$$s_1 = -\zeta + j\sqrt{1-\zeta^2} \omega_n$$
$$s_2 = -\zeta - j\sqrt{1-\zeta^2} \omega_n$$

The solution of the differential equation becomes

$$x = C_1 e^{-\zeta \omega_n t} + C_2 e^{-\zeta \omega_n t}$$

This equation can be rewritten as

$$x = e^{-\zeta \omega_n t} \left[ C_1 e^{j\sqrt{1-\zeta^2} \omega_n t} + C_2 e^{-j\sqrt{1-\zeta^2} \omega_n t} \right]$$  \hspace{1cm} (11)

Using the relationships

$$e^{i\theta} = \cos \theta + i \sin \theta$$
$$e^{-i\theta} = \cos \theta - i \sin \theta$$

Equation (11) can be written as

$$x = e^{-\zeta \omega_n t} \left[ C_1 \left\{ \cos \sqrt{1-\zeta^2} \omega_n t + j \sin \sqrt{1-\zeta^2} \omega_n t \right\} + C_2 \left\{ \cos \sqrt{1-\zeta^2} \omega_n t - j \sin \sqrt{1-\zeta^2} \omega_n t \right\} \right]$$

Or

$$x = e^{-\zeta \omega_n t} \left[ \left( C_1 + C_2 \right) \cos \sqrt{1-\zeta^2} \omega_n t + j(C_1 - C_2) \sin \sqrt{1-\zeta^2} \omega_n t \right]$$  \hspace{1cm} (12)

In equation (12), constants ($C_1+C_2$) and $j(C_1-C_2)$ are real quantities and hence, the equation can also be written as

$$x = e^{-\zeta \omega_n t} \left[ A \cos \sqrt{1-\zeta^2} \omega_n t + B \sin \sqrt{1-\zeta^2} \omega_n t \right]$$

Or

$$x = A e^{-\zeta \omega_n t} \left[ \sin \left( \sqrt{1-\zeta^2} \omega_n t + \phi \right) \right]$$  \hspace{1cm} (13)

The above equations represent oscillatory motion and the frequency of this motion is represented by

$$\omega_d = \sqrt{1-\zeta^2} \omega_n$$  \hspace{1cm} (14)

Where, $\omega_d$ is the damped natural frequency of the system. Constants $A_1$ and $\Phi_1$ are determined by applying initial conditions. The typical response curve of an under damped system is shown in Fig.6.
Applying initial conditions, \( x = X_0 \) at \( t = 0 \); and \( \dot{x} = 0 \) at \( t = 0 \), and finding constants \( A_1 \) and \( \Phi_1 \), equation (13) becomes

\[
x = \frac{X_0}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega \omega_0 t} \left[ \sin \sqrt{1 - \zeta^2} \omega_0 t + \tan^{-1} \left( \frac{\sqrt{1 - \zeta^2}}{\zeta} \right) \right]
\]  

(15)

The term \( \frac{X_0}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega \omega_0 t} \) represents the amplitude of vibration, which is observed to decay exponentially with time.
LOGARITHMIC DECREMENT

Referring to Fig.7, points A & B represent two successive peak points on the response curve of an under damped system. \(X_A\) and \(X_B\) represent the amplitude corresponding to points A & B and \(t_A\) & \(t_B\) represents the corresponding time.

We know that the natural frequency of damped vibration = \(\omega_d = \sqrt{1-\zeta^2} \omega_n\) rad/sec.

Therefore, \(f_d = \frac{\omega_d}{2\pi}\) cycles/sec

Hence, time period of oscillation = \(t_B - t_A = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2} \omega_n}\) sec \(\quad (16)\)

From equation (15), amplitude of vibration

\[X_A = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_d t_A}\]

\[X_B = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_d t_B}\]

Or,

\[\frac{X_A}{X_B} = e^{-\zeta \omega_d (t_B - t_A)} = e^{\zeta \omega_d (t_A - t_B)}\]

Using eqn. (16),

\[\frac{X_A}{X_B} = e^{\frac{2\pi \zeta}{\sqrt{1-\zeta^2}}}\]

Or,

\[\log e \frac{X_A}{X_B} = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}}\]
This is called logarithmic decrement. It is defined as the logarithmic value of the ratio of two successive amplitudes of an under damped oscillation. It is normally denoted by $\delta$.

Therefore, $\delta = \log_e \frac{X_A}{X_B} = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}}$ \hspace{1cm} (17)

This indicates that the ratio of any two successive amplitudes of an under damped system is constant and is a function of damping ratio of the system.

For small values of $\zeta$, $\delta = 2\pi \zeta$

If $X_0$ represents the amplitude at a particular peak and $X_n$ represents the amplitude after ‘n’ cycles, then, logarithmic decrement = $\delta = \log_e \frac{X_0}{X_1} = \log_e \frac{X_1}{X_2} = \ldots = \log_e \frac{X_{n-1}}{X_n}$

Adding all the terms, $n\delta = \log_e \frac{X_0 \times X_1 \times \ldots \times X_{n-1}}{X_1 \times X_2 \times \ldots \times X_n}$

Or, $\delta = \frac{1}{n} \log_e \frac{X_0}{X_n}$ \hspace{1cm} (18)
Solved problems

1) The mass of a spring-mass-dashpot system is given an initial velocity $5\omega_n$, where $\omega_n$ is the undamped natural frequency of the system. Find the equation of motion for the system, when (i) $\zeta = 2.0$, (ii) $\zeta = 1.0$, (i) $\zeta = 0.2$.

Solution:

Case (i) For $\zeta = 2.0$ – Over damped system

For over damped system, the response equation is given by

$$x = C_1 e^{\left[-\zeta \sqrt{\zeta^2 - 1}\right] \omega_n t} + C_2 e^{\left[-\zeta \sqrt{\zeta^2 - 1}\right] \omega_n t}$$

Substituting $\zeta = 2.0$, $x = C_1 e^{-0.27\omega_n t} + C_2 e^{-3.73\omega_n t}$ (a)

Differentiating, $\dot{x} = -0.27 \omega_n C_1 e^{-0.27\omega_n t} - 3.73 \omega_n C_2 e^{-3.73\omega_n t}$ (b)

Substituting the initial conditions

$$x = 0$$ at $t = 0$; and $\dot{x} = 5\omega_n$ at $t = 0$ in (a) & (b),

$$0 = C_1 + C_2$$ (c)

$$5\omega_n = -0.27 \omega_n C_1 - 3.73 \omega_n C_2$$ (d)

Solving (c) & (d), $C_1 = 1.44$ and $C_2 = -1.44$.

Therefore, the response equation becomes

$$x = 1.44\left(e^{-0.27\omega_n t} - e^{-3.73\omega_n t}\right)$$ (e)

Case (ii) For $\zeta = 1.0$ – Critically damped system

For critically damped system, the response equation is given by

$$x = (C_1 + C_2 t) e^{-\omega_n t}$$ (f)

Differentiating, $\dot{x} = -(C_1 + C_2 t) \omega_n e^{-\omega_n t} + C_2 e^{-\omega_n t}$ (g)

Substituting the initial conditions

$$x = 0$$ at $t = 0$; and $\dot{x} = 5\omega_n$ at $t = 0$ in (f) & (g),

$$C_1 = 0$$ and $C_2 = 5\omega_n$

Substituting in (f), the response equation becomes

$$x = (5\omega_n t) e^{-\omega_n t}$$ (h)

Case (iii) For $\zeta = 0.2$ – under damped system

For under damped system, the response equation is given by

$$x = A_t e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \omega_n t} \left[\sin \left(\sqrt{1-\zeta^2} \omega_n t + \phi_t\right)\right]$$

Substituting $\zeta = 0.2$, $x = A_0 e^{-0.2\omega_n t} \left[\sin \left(0.98\omega_n t + \phi_0\right)\right]$ (p)
Differentiating,
\[ \dot{x} = -0.2\omega_n A_1 e^{-0.2\omega_n t} \left[ \sin(0.98\omega_n t + \phi) \right] + 0.98\omega_n A_1 e^{-0.2\omega_n t} \cos(0.98\omega_n t + \phi) \] (q)

Substituting the initial conditions
\[ x = 0 \text{ at } t = 0; \text{ and } \dot{x} = 5\omega_n \text{ at } t = 0 \text{ in (p) & (q)}, \]
\[ A_1 \sin\phi = 0 \text{ and } A_1 \cos\phi = 5.1 \]

Solving, \[ A_1 = 5.1 \text{ and } \phi = 0 \]

Substituting in (p), the response equation becomes
\[ x = 5.1e^{-0.2\omega_n t} \left[ \sin(0.98\omega_n t) \right] \] (r)

2) A mass of 20kg is supported on two isolators as shown in fig.Q.2. Determine the undamped and damped natural frequencies of the system, neglecting the mass of the isolators.

Solution:

Equivalent stiffness and equivalent damping coefficient are calculated as

\[ \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{10000} + \frac{1}{3000} = \frac{13}{30000} \]
\[ \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{300} + \frac{1}{100} = \frac{4}{300} \]

Undamped natural frequency = \[ \omega_n = \frac{k_{eq}}{m} = \sqrt{\frac{30000}{13}} = \frac{10.74 \text{rad/sec}}{20} = 10.74 \text{rad/sec} \]

Damped natural frequency = \[ \omega_d = \sqrt{1-\zeta^2} \omega_n \]

\[ \zeta = \frac{C_{eq}}{2\sqrt{k_{eq}m}} = \frac{4}{2\times\sqrt{30000/13\times20}} = 0.1745 \]
3) A gun barrel of mass $500 \text{kg}$ has a recoil spring of stiffness $3,000,000 \text{ N/m}$. If the barrel recoils 1.2 meters on firing, determine,

(a) initial velocity of the barrel

(b) critical damping coefficient of the dashpot which is engaged at the end of the recoil stroke

(c) time required for the barrel to return to a position 50mm from the initial position.

Solution:

(a) Strain energy stored in the spring at the end of recoil:

$$P = \frac{1}{2}kx^2 = \frac{1}{2} \times 300000 \times 1.2^2 = 216000 \text{N} - m$$

Kinetic energy lost by the gun barrel:

$$T = \frac{1}{2}mv^2 = \frac{1}{2} \times 500 \times v^2 = 250v^2$$

where $v = \text{initial velocity of the barrel}$

Equating kinetic energy lost to strain energy gained, ie $T = P$,

$$250v^2 = 216000$$

$$v = 29.39 \text{m/s}$$

(b) Critical damping coefficient $= C_c = 2\sqrt{km} = 2\sqrt{300000 \times 500} = 24495 \text{N – sec/m}$

(c) Time for recoiling of the gun (undamped motion):

Undamped natural frequency $= \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{300000}{500}} = 49.24 \text{rad/s}$

Time period $= \tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{49.24} = 0.259 \text{sec}$

Time of recoil $= \frac{\tau}{4} = \frac{0.259}{4} = 0.065 \text{sec}$

Time taken during return stroke:

Response equation for critically damped system $= x = (C_1 + C_2t)e^{-\omega_d t}$

Differentiating, $\dot{x} = C_2e^{-\omega_d t} - (C_1 + C_2t)\omega_d e^{-\omega_d t}$

Applying initial conditions, $x = 1.2$, at $t = 0$ and $\dot{x} = 0$ at $t = 0$,

$C_1 = 1.2$, & $C_2 = 29.39$

Therefore, the response equation $= x = (1.2 + 29.39t)e^{-49.24t}$

When $x = 0.05 \text{m}$, by trial and error, $t = 0.20 \text{sec}$

Therefore, total time taken = time for recoil + time for return $= 0.065 + 0.20 = 0.265 \text{sec}$

The displacement – time plot is shown in the following figure.
4) A 25 kg mass is resting on a spring of 4900 N/m and dashpot of 147 N-se/m in parallel. If a velocity of 0.10 m/sec is applied to the mass at the rest position, what will be its displacement from the equilibrium position at the end of first second?

Solution:

The above figure shows the arrangement of the system.

Critical damping coefficient = $c_c = 2m\omega_n$

Where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4900}{25}} = 14 \text{ rad/s}$

Therefore, $c_c = 2 \times 25 \times 14 = 700 \text{ N - sec/m}$

Since $C < C_c$, the system is under damped and $\zeta = \frac{c}{c_c} = \frac{147}{700} = 0.21$
Hence, the response equation is \[ x = A_t e^{-\zeta \omega_n t} \left[ \sin \left( \sqrt{1 - \zeta^2} \omega_n t + \phi \right) \right] \]

Substituting \( \zeta \) and \( \omega_n \), \[ x = A_t e^{-0.2144t} \left[ \sin \left( \sqrt{1 - 0.21^2} \cdot 14t + \phi \right) \right] \]
\[ x = A_t e^{-2.94t} \left[ \sin(13.7t + \phi) \right] \]

Differentiating, \[ \dot{x} = -2.94A_t e^{-2.94t} \left[ \sin(13.7t + \phi) \right] + 13.7A_t e^{-2.94t} \cos(13.7t + \phi) \]

Applying the initial conditions, \( x = 0 \), at \( t = 0 \) and \( \dot{x} = 0.10 \text{ m/s} \) at \( t = 0 \)
\[ \Phi_1 = 0 \]

\[ 0.10 = -2.94A_t \left[ \sin(\phi) \right] + 13.7A_t \cos(\phi) \]

Since, \( \Phi_1 = 0 \), \( 0.10 = 13.7 A_1 \); \( A_1 = 0.0073 \)

Displacement at the end of 1 second = \( x = 0.0073 e^{-2.94t} \left[ \sin(13.7t) \right] = 3.5 \times 10^{-4} \text{ m} \)

5) A rail road bumper is designed as a spring in parallel with a viscous damper. What is the bumper’s damping coefficient such that the system has a damping ratio of 1.25, when the bumper is engaged by a rail car of 20000 kg mass. The stiffness of the spring is 2E5 N/m. If the rail car engages the bumper, while traveling at a speed of 20m/s, what is the maximum deflection of the bumper?

\[ \text{Solution: Data} = m = 20000 \text{ kg}; k = 200000 \text{ N/m}; \zeta = 1.25 \]

Critical damping coefficient = \[ c_c = 2 \times \sqrt{m \times k} = 2 \times \sqrt{20000 \times 200000} = 1.24 \times 10^5 \text{ N/sec/m} \]

Damping coefficient \( C = \zeta \times C_c = 1.25 \times 1.24 \times 10^5 = 1.58 \times 10^5 \text{ N/sec/m} \)

Undamped natural frequency = \( \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200000}{20000}} = 3.16 \text{ r/s} \)

Since \( \zeta = 1.25 \), the system is over damped.

For over damped system, the response equation is given by
\[ x = C_1 e^{-\zeta \omega_n t} + \frac{C_2}{\sqrt{\zeta^2 - 1}} e^{-\omega_n t} \]

Substituting \( \zeta = 1.25 \), \[ x = C_1 e^{-0.5 \omega_n t} + C_2 e^{-2.0 \omega_n t} \] (a)
Differentiating,

\[ \dot{x} = -0.5 \omega^2 C_1 e^{-0.5 \omega t} - 2.0 \omega C_2 e^{-2 \omega t} \]  

Substituting the initial conditions

\[ x = 0 \text{ at } t = 0; \text{ and } \dot{x} = 20\text{ m/s} \text{ at } t = 0 \] in (a) & (b),

\[ 0 = C_1 + C_2 \quad \text{(c)} \]
\[ 20 = -0.5 \omega C_1 - 2.0 \omega C_2 \quad \text{(d)} \]

Solving (c) & (d), \( C_1 = 4.21 \) and \( C_2 = -4.21 \)

Therefore, the response equation becomes

\[ x = 4.21 \left( e^{-1.58 \omega t} - e^{-6.32 \omega t} \right) m \quad \text{(e)} \]

The time at which, maximum deflection occurs is obtained by equating velocity equation to zero.

\[ \dot{x} = -0.5 \omega C_1 e^{-0.5 \omega t} - 2.0 \omega C_2 e^{-2 \omega t} = 0 \]

\[ -6.65 e^{-1.58 \omega t} + 26.61 e^{-6.32 \omega t} = 0 \]

Solving the above equation, \( t = 0.292 \) secs.

Therefore, maximum deflection at \( t = 0.292 \) secs,

Substituting in (e), \( x = 4.21 \left( e^{-1.58 \omega t} - e^{-6.32 \omega t} \right) m \), \( = 1.99 \) m.

6) A disc of a torsional pendulum has a moment of inertia of \( 6E^{-2} \text{ kg-m}^2 \) and is immersed in a viscous fluid. The shaft attached to it is \( 0.4 \text{ m} \) long and \( 0.1 \text{ m} \) in diameter. When the pendulum is oscillating, the observed amplitudes on the same side of the mean position for successive cycles are \( 9^\circ, 6^\circ \) and \( 4^\circ \). Determine (i) logarithmic decrement (ii) damping torque per unit velocity and (iii) the periodic time of vibration. Assume \( G = 4.4E10 \text{ N/m}^2 \), for the shaft material.

Solution:

The above figure shows the arrangement of the system.

(i) Logarithmic decrement = \( \delta = \log_e \frac{9}{6} = \log_e \frac{6}{4} = 0.405 \)

(ii) The damping torque per unit velocity = damping coefficient of the system ‘C’.
We know that logarithmic decrement = \( \delta = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} \), rearranging which, we get

Damping factor \( \zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.405}{\sqrt{4\pi^2 + 0.405^2}} = 0.0645 \)

Also, \( \zeta = \frac{C}{C_c} \), where, critical damping coefficient = \( C_c = 2\sqrt{k_J} \)

Torsional stiffness = \( k_J = \frac{G l}{l} = \frac{G \pi d^4}{32} = \frac{4.4 \times 10^{10} \pi \times 0.1^4}{32} = 1.08 \times 10^6 N - m / rad \)

Critical damping coefficient = \( C_c = 2\sqrt{k_J} = 2\sqrt{1.08 \times 10^6 \times 0.06} = 509 N - m / rad \)

Damping coefficient of the system = \( C = C_c \times \zeta = 509 \times 0.0645 = 32.8 N - m / rad \)

(iii) Periodic time of vibration = \( \tau = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} \)

Where, undamped natural frequency = \( \omega_n = \sqrt{\frac{k_J}{J}} = \sqrt{\frac{1.08 \times 10^6}{0.06}} = 4242.6 rad / sec \)

Therefore, \( \tau = \frac{2\pi}{4242.6 \times \sqrt{1 - 0.0645^2}} = 0.00148 sec \)

7) A mass of 1 kg is to be supported on a spring having a stiffness of 9800 N/m. The damping coefficient is 5.9 N-sec/m. Determine the natural frequency of the system. Find also the logarithmic decrement and the amplitude after three cycles if the initial displacement is 0.003m.

\[ m = 1 \text{ kg} \]
\[ k = 9800 \text{ N/m} \]
\[ C = 5.9 \text{ N-sec/m} \]

Solution:

Undamped natural frequency = \( \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9800}{1}} = 99 \text{ r/s} \)

Damped natural frequency = \( \omega_d = \sqrt{1 - \zeta^2} \omega_n \)

Critical damping coefficient = \( c_c = 2 \times m \times \omega_n = 2 \times 1 \times 99 = 198 N - sec / m \)

Damping factor = \( \zeta = \frac{c}{c_c} = \frac{5.9}{198} = 0.03 \)

Hence damped natural frequency = \( \omega_d = \sqrt{1 - \zeta^2} \omega_n = \sqrt{1 - 0.03^2} \times 99 = 98.99 \text{ rad / sec} \)
Logarithmic decrement \( \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\times\pi\times 0.03}{\sqrt{1-0.03^2}} = 0.188 \)

Also, \( \delta = \frac{1}{n} \log_e \frac{X_0}{X_n} \); if \( x_0 = 0.003 \),

then, after 3 cycles, \( \delta = \frac{1}{n} \log_e \frac{X_0}{X_n} i.e. 0.188 = \frac{1}{3} \times \log_e \frac{0.003}{X_3} \)

i.e. \( X_3 = \frac{0.003}{e^{0.188}} = 1.71 \times 10^{-3} m \)

8) The damped vibration record of a spring-mass-dashpot system shows the following data.
Amplitude on second cycle = 0.012m; Amplitude on third cycle = 0.0105m;
Spring constant \( k = 7840\) N/m; Mass \( m = 2\)kg. Determine the damping constant, assuming it to be viscous.

Solution:

Here, \( \delta = \log_e \frac{X_2}{X_3} = \log_e \frac{0.012}{0.0105} = 0.133 \)

Also, \( \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \), rearranging, \( \zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.133}{\sqrt{4\pi^2 + 0.133^2}} = 0.021 \)

Critical damping coefficient = \( c_c = 2 \times \sqrt{m \times k} = 2 \times \sqrt{2 \times 7840} = 250.4 N - sec/ m \)

Damping coefficient \( C = \zeta \times C_c = 0.021 \times 250.4 = 5.26 N - sec/ m \)

9) A mass of 2kg is supported on an isolator having a spring scale of 2940 N/m and viscous damping. If the amplitude of free vibration of the mass falls to one half its original value in 1.5 seconds, determine the damping coefficient of the isolator.

![Diagram](m = 2 kg, k = 2940 N/m, C)

Solution:

Undamped natural frequency = \( \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2940}{2}} = 38.34 r/s \)
Critical damping coefficient = \( c_c = 2 \times m \times \omega_n = 2 \times 2 \times 38.34 = 153.4 N \text{ sec/m} \)

Response equation of under damped system = 
\[ x = A_t e^{-\zeta \omega_n t} \left[ \sin \left( \sqrt{1 - \zeta^2} \omega_n t + \phi \right) \right] \]

Here, amplitude of vibration = \( A_t e^{-\zeta \omega_n t} \)

If amplitude = \( X_0 \) at \( t = 0 \), then, at \( t = 1.5 \text{ sec} \), amplitude = \( \frac{X_0}{2} \)

I.e., \( A_t e^{-\zeta \omega_n \times 0} = X_0 \) or \( A_t = X_0 \)

Also, \( A_t e^{-\zeta \omega_n \times 1.5} = \frac{X_0}{2} \) or \( X_0 \times e^{-\zeta \times 38.34 \times 1.5} = \frac{X_0}{2} \) or \( e^{-\zeta \times 38.34 \times 1.5} = \frac{1}{2} \)

I.e., \( e^{\zeta \times 38.34 \times 1.5} = 2 \), taking log, \( \zeta \times 38.34 \times 1.5 = 0.69 \therefore \zeta = 0.012 \)

Damping coefficient \( C = \zeta \times C_c = 0.012 \times 153.4 = 1.84 N \text{ sec/m} \)