Goertzel’s Algorithm

Introduction:
Frequency Analysis
- Standard frequency analysis requires transforming time-domain signal to frequency domain and studying Spectrum of the signal. This is done through DFT computation.
- N-point DFT computation results in N frequency components
- DFT computation through FFT requires $N/2 \log_2 N$ complex multiplications and $N \log_2 N$ additions.
- In certain applications not all N frequency components need to be computed (an application will be discussed)
- If the desired number of values of the DFT is less than $2 \log_2 N$ than direct computation of the desired values is more efficient than FFT based computation.

Example: DTMF – Dual Tone Multifrequency
- This is known as touch-tone/speed/electronic dialing
- Pressing of each button generates a unique set of two-tone signals, called DTMF signals.
- These signals are processed at exchange to identify the number pressed by determining the two associated tone frequencies.
- Seven frequencies are used to code the 10 decimal digits and two special characters (4x3 array)
In this application frequency analysis requires determination of possible seven (eight) DTMF fundamental tones and their respective second harmonics.

For an 8 kHz sampling freq, the best value of the DFT length \( N \) to detect the eight fundamental DTMF tones has been found to be 205.

Not all 205 freq components are needed here, instead only those corresponding to key frequencies are required.

FFT algorithm is not effective and efficient in this application.

The direct computation of the DFT can be formulated as a linear filtering operation on the input data sequence.

This algorithm is known as **Goertzel Algorithm**.

This algorithm exploits periodicity property of the phase factor

\[
X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk}
\] (1)

Since \( W_N^{-kn} \) is equal to 1, multiplying both sides of the equation by this results in;

\[
X(k) = W_N^{-kn} \sum_{m=0}^{N-1} x(m)W_N^{mk} = \sum_{m=0}^{N-1} x(m)W_N^{-k(N-m)}
\] (2)

This is in the form of a convolution

\[
y_k(n) = x(n) * h_k(n)
\]

\[
y_k(n) = \sum_{m=0}^{N-1} x(m)W_N^{-k(n-m)}
\] (3)

\[
h_k(n) = W_N^{-k_n}u(n)
\] (4)

Where \( y_k(n) \) is the output of a filter which has impulse response of \( h_k(n) \) and input \( x(n) \).
The output of the filter at \( n = N \) yields the value of the DFT at the freq \( \omega_k = 2\pi k/N \).

The filter has frequency response given by
\[
H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}} \quad (6)
\]

The above form of filter response shows it has a pole on the unit circle at the frequency \( \omega_k = 2\pi k/N \).

Entire DFT can be computed by passing the block of input data into a parallel bank of \( N \) single-pole filters (resonators).

**Difference Equation implementation of filter:**
From the frequency response of the filter (eq 6) we can write the following difference equation relating input and output;
\[
H_k(z) = \frac{Y_k(z)}{X(z)} = \frac{1}{1 - W_N^{-k} z^{-1}}
\]
\[
y_k(n) = W_N^{-k} y_k(n-1) + x(n) \quad y_k(-1) = 0 \quad (7)
\]

The desired output is \( X(k) = y_k(n) \) for \( k = 0,1,...,N-1 \).

The phase factor appearing in the difference equation can be computed once and stored.

The form shown in eq (7) requires complex multiplications which can be avoided doing suitable modifications (divide and multiply by \( 1 - W_N^{-k} z^{-1} \))

The frequency response of the filter can be alternatively expressed as
\[
H_k(z) = \frac{1 - W_N^{-k} z^{-1}}{1 - 2 \cos(2\pi k/N) z^{-1} + z^{-2}} \quad (8)
\]

The direct form realization of the above is given by
\[
v_k(n) = 2 \cos(2\pi k/N) v_k(n-1) - v_k(n-2) + x(n) \quad (9)
\]
\[
y_k(n) = v_k(n) - W_N^k v_k(n-1) \quad v_k(-1) = v_k(-2) = 0 \quad (10)
\]
• The recursive relation in (9) is iterated for 
n = 0,1,……N, but the equation in (10) is computed only once at time n =N  
• Each iteration requires one real multiplication and two additions.  
• Thus, for a real input sequence x(n) this algorithm requires (N+1) real multiplications to yield X(k) and X(N-k) (this is due to symmetry)  
• Going through the Goertzel algorithm it is clear that this algorithm is useful only when M out of N DFT values need to be computed where M≤ 2log₂N, Otherwise, the FFT algorithm is more efficient method.

**Chirp z- Transform**

**Situations where DFT computation through FFT is not preferable:**

• Computation of DFT is equivalent to samples of the z-transform of a finite-length sequence at equally spaced points around the unit circle.  
• The efficient computation of DFT through FFT requires N to be a highly composite number  
• Many a times we may need samples of z-transform on contours other than unit circle or dense set of frequency samples over a small region of unit circle.

Consider these problems:

• Obtain samples of z-transform on a circle of radius ‘a’ which is concentric to unit circle  
  Soln: The sequence need to be multiplied a⁻ⁿ  
• 128 samples needed between frequencies  
  ω = -π/8 to +π/8 from a 128 point sequence  
  soln: 1024- point FFT computation where the given 128-point point sequence is appended with 896 zeros. But only 128 frequencies out of 1024 needed, hence wastage of computations.
Chirp z-transform is the alternative.

Chirp z-transform is defined as:

$$X(z_k) = \sum_{n=0}^{N-1} x(n) z_k^{-n} \quad k = 0,1,\ldots,L-1$$

(11)

Where $z_k$ is generalized contour.

- $Z_k$ is the set of points in the z-plane falling on an arc which begins at some point $z_0$ and spirals either in toward the origin or out away from the origin such that the points $\{z_k\}$ are defined as,

$$z_k = r_0 e^{j\theta_0} (R_0 e^{j\theta_k})^k \quad k = 0,1,\ldots,L-1$$

(12)

Note that,

- if $R_0 < 1$ the points fall on a contour that spirals toward the origin

- if $R_0 > 1$ the contour spirals away from the origin

- If $R_0=1$ the contour is a circular arc of radius

- If $r_0=1$ and $R_0=1$ the contour is an arc of the unit circle.

**Additionally this contour allows one to compute the freq content of the sequence $x(n)$ at a dense set of $L$ frequencies in the range covered by the arc without having to compute a large DFT (i.e., a DFT of the sequence $x(n)$ padded with many zeros to obtain the desired resolution in freq.)

- If $r_0= R_0=1$ and $\theta_0=0 \Phi_0=2\pi/N$ and $L = N$ the contour is the entire unit circle similar to the standard DFT.
Substituting this value of $z_k$ in the expression of $X(z_k)$

$$X(z_k) = \sum_{n=0}^{N-1} x(n)z_k^{-n} = \sum_{n=0}^{N-1} x(n)(r_0e^{i\theta_0})^{-n}W^{-nk}$$  \hspace{1cm} (13)

where

$$W = R_0e^{i\phi_0}$$  \hspace{1cm} (14)

**Expressing computation of $X(z_k)$ as linear filtering operation:**

By substitution of

$$nk = \frac{1}{2}(n^2 + k^2 - (k-n)^2)$$  \hspace{1cm} (15)

we can express $X(z_k)$ as

$$X(z_k) = W^{-k^2/2}y(k) = y(k) / h(k) \quad k = 0, 1, \ldots, L-1$$  \hspace{1cm} (16)

Where

$$h(n) = W^{n^2/2} \quad g(n) = x(n)(r_0e^{i\phi_0})^{-n}W^{-n^2/2}$$

$$y(k) = \sum_{n=0}^{N-1} g(n)h(k-n)$$  \hspace{1cm} (17)

both $g(n)$ and $h(n)$ are complex valued sequences
Why it is called Chirp z-transform?
if \( R_0 = 1 \), then seq \( h(n) \) has the form of complex exponential with argument \( \omega n = n^2 \Phi_0/2 = (n \Phi_0/2) n \). The quantity \( (n \Phi_0/2) \) represents the freq of the complex exponential signal, which increases linearly with time. Such signals are used in radar systems are called chirp signals. Hence the name chirp z-transform.

Evaluation of linear convolution in eq (17)
- Can be done efficiently with FFT
- \( g(n) \) is finite length seq of length \( N \) and \( h(n) \) is of infinite duration, but fortunately only a portion of \( h(n) \) is required to compute \( L \) values of \( X(z) \)
- Since convolution is via FFT, it is circular convolution of the \( N \)-point seq \( g(n) \) with an \( M \)-point section of \( h(n) \) where \( M > N \)
- The concepts used in over lap –save method can be used
- While circular convolution is used to compute linear convolution of two sequences we know the initial \( N-1 \) points contain aliasing and the remaining points are identical to the result that would be obtained from a linear convolution of \( h(n) \) and \( g(n) \)
- In view of this the DFT size selected is \( M = L+N-1 \) which would yield \( L \) valid points and \( N-1 \) points corrupted by aliasing.
- The section of \( h(n) \) considered is for \( -(N-1) \leq n \leq (L-1) \) yielding total length \( M \) as defined
- The portion of \( h(n) \) can be defined in many ways
  - \( h_1(n) = h(n-N+1) \) \( n = 0,1,\ldots,M-1 \)
  - Compute \( H_1(k) \) and \( G(k) \) to obtain \( Y_1(k) = G(K)H_1(k) \)
  - Application of IDFT will give \( y_1(n) \), for \( n = 0,1,\ldots,M-1 \)
• The starting N-1 are discarded and desired values are $y_1(n)$ for $N-1 \leq n \leq M-1$ which corresponds to the range $0 \leq n \leq L-1$ i.e.,
• $y(n) = y_1(n+N-1)$, $n=0,1,2,\ldots,L-1$
• Alternatively $h_2(n)$ can be defined as
  
  $$h_2(n) = h(n) \quad 0 \leq n \leq L-1$$
  $$h(n-(N+L-1)) \quad L \leq n \leq M-1$$

  
  • Compute $Y_2(k) = G(K)H_2(k)$
  • The desired values of $y_2(n)$ are in the range $0 \leq n \leq L-1$ i.e.,
  
  $$y(n) = y_2(n) \quad n=0,1,\ldots,L-1$$
  • Finally, the complex values $X(z_k)$ are computed by dividing $y(k)$ by $h(k)$
  
  For $k =0,1,\ldots,L-1$

**Computational complexity**
- In general, the computational complexity of CZT is of the order of $M \log_2 M$ complex multiplications
- This should be compared with $N.L$ which is required for direct evaluation
- If $L$ is small direct evaluation is more efficient otherwise if $L$ is large then CZT is more efficient.

**Advantages of CZT**
- Not necessary to have $N = L$
- Neither $N$ or $L$ need to be highly composite
- The samples of Z transform are taken on a more general contour that includes the unit circle as a special case.

**Example to understand utility of CZT algorithm in freq analysis**
(ref: DSP by Oppenheim Schaffer)
- CZT is used in this application to sharpen the resonances by evaluating the z-transform off the unit circle.
- Signal to be analyzed is a synthetic speech signal generated by exciting a five-pole system with a periodic impulse train. The system was simulated to correspond to a sampling freq of 10khz.
- The poles are located at center freqs of $270,2290,3010,3500 \& 4500$ Hz with bandwidth of $30, 50, 60, 87 \& 140$ Hz respectively.
The first two spectra correspond to spiral contours outside the unit circle with a resulting broadening of the resonance peaks.

$|w| = 1$ corresponds to evaluating $z$-transform on the unit circle.

The last two choices correspond to spiral contours which spiral inside the unit circle and close to the pole locations resulting in a sharpening of resonance peaks.

**Implementation of CZT in hardware to compute the DFT signals**

- DFT computation requires $r_0 = R_0 = 1$, $\theta_0 = 0$
- $\Phi_0 = 2\pi/N$ and $L = N$
- The cosine and sine sequences in $h(n)$ needed for pre multiplication and post multiplication are usually stored in a ROM
- If only magnitude is of DFT is desired, the post multiplication are unnecessary
- In this case $|X(\omega_k)| = |y(k)|$ for $k = 0, 1, \ldots, N-1$
Design of Digital Filters

- Two important classes of filters based on impulse response type are
  - **Finite Impulse Response (FIR)**
  - **Infinite Impulse Response (IIR)**

The filter can be expressed as:

**System function representation**;

\[ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \] (1)

**Difference Equation representation**;

\[ \sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \] (2)

- Each of this form allows various methods of implementation
- The eq (2) can be viewed as a computational procedure (an algorithm) for determining the output sequence \( y(n) \) of the system from the input sequence \( x(n) \)
- Different realizations possible with different arrangements of eq (2)

**Filter Design Issues:**
- Realizability
- Stability
- Sharp Cutoff Characteristics
- Minimum order
- Generalized procedure
- Linear phase characteristics

**Issues considered for filter implementation:**
- Simple design
- Structuredness – modularity
- Generalization of design – any filter type one design method
- Cost of implementation
- Software/Hardware realization
Features of IIR:
- Output is a function of past output, present and past input’s.
- Recursive nature.
- Has at least one pole (in general, poles and zeros).
- Sharp cutoff can be achieved with minimum order.
- Difficult to have linear phase over full range of frequencies.
- Typical design procedure is analog design then conversion from analog to digital.

Features of FIR:
- Inherently stable.
- Linear phase characteristics possible.
- Simple implementation – both recursive and non-recursive structures possible.
- Free of limit cycle oscillations when implemented on a finite-word length digital system.

Disadvantages:
- Sharp cutoff at the cost of higher order.
- Higher order leading to more delay, more memory and higher cost of implementation.

Importance of Linear Phase:
The group delay is defined as
\[ \tau_g = -\frac{d\theta(\omega)}{d\omega} \]
- Nonlinear phase results in different frequencies experiencing different delay and arriving at different time at the receiver.
- This creates problems with speech processing and data communication applications.

Understanding simple FIR filtering operations:
1. Unity Gain Filter
   \[ y(n) = x(n) \]
2. Constant Gain Filter
   \[ y(n) = Kx(n) \]
3. Unit Delay Filter
   \[ y(n) = x(n-1) \]
4. Two-term Difference Filter
   \[ y(n) = x(n) - x(n-1) \]
5. Two-term Average Filter
   \[ y(n) = 0.5(x(n) + x(n-1)) \]
6. Three-term Average Filter (3-point moving average filter)
   \[ y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)] \]
7. Central Difference Filter
   \[ y(n) = \frac{1}{2}[x(n) - x(n-2)] \]
• Order of the filter is the number of previous inputs used to compute the current output
• Filter coefficients are the numbers associated with each of the terms x(n), x(n-1), etc.
The table below shows order and filter coefficients of above simple filter types:

<table>
<thead>
<tr>
<th>Ex.</th>
<th>order</th>
<th>a0</th>
<th>a1</th>
<th>a2</th>
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<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>K</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4(HP)</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>5(LP)</td>
<td>1</td>
<td>1/2</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>6(LP)</td>
<td>2</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>7(HP)</td>
<td>2</td>
<td>1/2</td>
<td>0</td>
<td>-1/2</td>
</tr>
</tbody>
</table>

**Design of FIR filters:**

**Symmetric and Antisymmetric FIR filters giving out Linear Phase characteristics:**
- Symmetry in filter impulse response will ensure Linear phase

An FIR filter of length M with i/p x(n) & o/p y(n) is described by the difference equation:

\[
y(n) = b_0 x(n) + b_1 x(n-1) + \ldots + b_{M-1} x(n-(M-1)) = \sum_{k=0}^{M-1} b_k x(n-k) \quad \text{-(1)}
\]

Alternatively, it can be expressed in convolution form

\[
y(n) = \sum_{k=0}^{M-1} h(k) x(n-k) \quad \text{-(2)}
\]

i.e. \(b_k = h(k), k=0,1,\ldots, M-1\)

Filter is also characterized by

\[
H(z) = \sum_{k=0}^{M-1} h(k) z^{-k} \quad \text{-(3)}
\]

polynomial of degree M-1 in the variable \(z^{-1}\). The roots of this polynomial constitute zeros of the filter.

An FIR filter has linear phase if its unit sample response satisfies the condition

\[
h(n) = \pm h(M-1-n) \quad n=0,1,\ldots, M-1 \quad \text{-(4)}
\]
Incorporating this symmetry & anti symmetry condition in eq 3 we can show linear phase chas of FIR filters.

\[ H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \ldots + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)} \]

If \( M \) is odd

\[ H(z) = h(0) + h(1)z^{-1} + \ldots + h\left(\frac{M-1}{2}\right)z^{-\left(\frac{M-1}{2}\right)} + h\left(\frac{M+1}{2}\right)z^{-\left(\frac{M+1}{2}\right)} + h\left(\frac{M+3}{2}\right)z^{-\left(\frac{M+3}{2}\right)} + \ldots
\]

\[ + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)} \]

\[ = z^{-\left(\frac{M-1}{2}\right)} \left[ h(0)z^{\left(\frac{M-1}{2}\right)} + h(1)z^{\left(\frac{M-3}{2}\right)} + \ldots + h\left(\frac{M-1}{2}\right)z^{-1} + h\left(\frac{M+1}{2}\right)z^{-2} + \ldots + h(M-1)z^{-\left(\frac{M-1}{2}\right)} \right] \]

Applying symmetry conditions for \( M \) odd

\[ h(0) = \pm h(M-1) \]
\[ h(1) = \pm h(M-2) \]

\[ \ldots \]

\[ h\left(\frac{M-1}{2}\right) = \pm h\left(\frac{M-1}{2}\right) \]
\[ h\left(\frac{M+1}{2}\right) = \pm h\left(\frac{M-3}{2}\right) \]

\[ \ldots \]
\[ h(M-1) = \pm h(0) \]

\[ H(z) = z^{-\left(\frac{M-1}{2}\right)} \left[ h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left\{ z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2} \right\} \right] \]

Similarly for \( M \) even

\[ H(z) = z^{-\left(\frac{M-1}{2}\right)} \left[ \sum_{n=0}^{\frac{M-1}{2}} h(n)\left\{ z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2} \right\} \right] \]

Frequency response:
If the system impulse response has symmetry property (i.e., \( h(n) = h(M-1-n) \)) and \( M \) is odd
\[
H(e^{j\omega}) = e^{j\theta(\omega)} | H_r(e^{j\omega}) | \text{ where}
\]

\[
H_r(e^{j\omega}) = \left[ h\left(\frac{M-1}{2}\right) + 2 \sum_{n=0}^{M-3} h(n) \cos \omega \left(\frac{M-1}{2} - n\right) \right]
\]

\[
\theta(\omega) = -\left(\frac{M-1}{2}\right)\omega \quad \text{if } |H_r(e^{j\omega})| \geq 0
\]

\[
= -\left(\frac{M-1}{2}\right)\omega + \pi \quad \text{if } |H_r(e^{j\omega})| \leq 0
\]

In case of M even the phase response remains the same with magnitude response expressed as

\[
H_r(e^{j\omega}) = \left[ \frac{M}{2} \sum_{n=0}^{M-1} h(n) \cos \omega \left(\frac{M-1}{2} - n\right) \right]
\]

If the impulse response satisfies anti symmetry property (i.e., \(h(n) = -h(M-1-n)\)) then for M odd we will have

\[
h\left(\frac{M-1}{2}\right) = -h\left(\frac{M-1}{2}\right) \text{ i.e., } h\left(\frac{M-1}{2}\right) = 0
\]

\[
H_r(e^{j\omega}) = \left[ 2 \sum_{n=0}^{M-3} h(n) \sin \omega \left(\frac{M-1}{2} - n\right) \right]
\]

If M is even then,

\[
H_r(e^{j\omega}) = \left[ 2 \sum_{n=0}^{M-1} h(n) \sin \omega \left(\frac{M-1}{2} - n\right) \right]
\]

In both cases the phase response is given by

\[
\theta(\omega) = -\left(\frac{M-1}{2}\right)\omega + \pi / 2 \quad \text{if } |H_r(e^{j\omega})| \geq 0
\]

\[
= -\left(\frac{M-1}{2}\right)\omega + 3\pi / 2 \quad \text{if } |H_r(e^{j\omega})| \leq 0
\]

Which clearly shows presence of Linear Phase characteristics.
Comments on filter coefficients:

• The number of filter coefficients that specify the frequency response is \((M+1)/2\) when
  \(M\) is odd and \(M/2\) when \(M\) is even in case of symmetric conditions
• In case of impulse response antisymmetric \(h(M-1/2)=0\) so that there are \((M-1/2)\) filter
  coefficients when \(M\) is odd and \(M/2\) coefficients when \(M\) is even

Choice of Symmetric and antisymmetric unit sample response

• If \(h(n)=-h(M-1-n)\) and \(M\) is odd, \(H_r(w)\) implies that \(H_r(0)=0\) \& \(H_r(\pi)=0\), consequently
  not suited for lowpass and highpass filter.
• Similarly if \(M\) is even \(H_r(0)=0\) hence not used for low pass filter
• Hence antisymmetry condition is not generally used
• Symmetry condition \(h(n)=h(M-1-n)\) yields a linear-phase FIR filter with non zero
  response at \(w=0\) if desired.

Zeros of Linear Phase FIR Filters

Consider the filter system function

\[
H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}
\]

Expanding this equation

\[
H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \ldots + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)}
\]

Since for Linear – phase we need

\[
h(n) = h(M-1-n) \quad i.e.,
\]

\[
h(0) = h(M-1); h(1) = h(M-2); \ldots h(M-1) = h(0);
\]

\[
H(z) = h(M-1) + h(M-2)z^{-1} + \ldots + h(1)z^{-(M-2)} + h(0)z^{-(M-1)}
\]

\[
H(z) = z^{-(M-1)}[h(M-1)z^{(M-1)} + h(M-2)z^{(M-2)} + \ldots + h(1)z + h(0)]
\]

\[
H(z) = z^{-(M-1)} \sum_{n=0}^{M-1} h(n)(z^{-1})^n = z^{-(M-1)}H(z^{-1})
\]

This shows that if \(z = z_1\) is a zero then \(z = z_1^{-1}\) is also a zero

The different possibilities:

• If \(z_1 = 1\) then \(z_1 = z_1^{-1} = 1\) is also a zero implying it is one zero
• If the zero is real and \(|z|<1\) then we have pair of zeros
• If zero is complex and \(|z|=1\) then and we again have pair of complex zeros.
• If zero is complex and \(|z|\neq 1\) then and we have two pairs of complex zeros
The plot above shows distribution of zeros for a Linear – phase FIR filter.

Methods of designing FIR filters:

- Fourier series based method
- Window based method
- Frequency sampling method

Design of Linear Phase FIR filter based on Fourier Series method:

**Motivation**: Since the desired freq response $H_d(e^{j\omega})$ is a periodic function in $\omega$ with period $2\pi$, it can be expressed as Fourier series expansion

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)e^{-j\omega n}$$

where $h_d(n)$ are fourier series coefficients

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n} d\omega$$

This expansion results in impulse response coefficients which are infinite in duration and non causal.

- It can be made finite duration by truncating the infinite length
• The linear phase can be obtained by introducing symmetric property in the filter impulse response, i.e., \( h(n) = h(-n) \)
• It can be made causal by introducing sufficient delay (depends on filter length)

**Stepwise procedure:**
- From the desired freq response using inverse FT relation obtain \( h_d(n) \)
- Truncate the infinite length of the impulse response to finite length with (assuming \( M \) odd)
  \[
  h(n) = h_d(n) \quad \text{for } -\frac{(M-1)}{2} \leq n \leq \frac{(M-1)}{2}
  \]
  \[
  = 0 \quad \text{otherwise}
  \]
- Introduce \( h(n) = h(-n) \) for linear phase characteristics
- Write the expression for \( H(z) \); this is non-causal realization
- To obtain causal realization \( H'(z) = z^{-\frac{(M-1)}{2}} H(z) \)

**Prob:** Design an ideal bandpass filter with a frequency response:

\[
H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4}
\]

\[
= 0 \quad \text{otherwise}
\]

Find the values of \( h(n) \) for \( M = 11 \) and plot the frequency response.

\[
h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega
\]

\[
= \frac{1}{2\pi} \left[ \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{j\omega n} d\omega + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{j\omega n} d\omega \right]
\]

\[
= \frac{1}{m} \left[ \sin \frac{3\pi}{4} n - \sin \frac{\pi}{4} n \right] \quad -\infty \leq n \leq \infty
\]

truncating to 11 samples we have \( h(n) = h_d(n) \) for \( |n| \leq 5 \)

\[
= 0 \quad \text{otherwise}
\]
• For \( n = 0 \) the value of \( h(n) \) is separately evaluated from the basic integration
\[ h(0) = 0.5 \]
• Other values of \( h(n) \) are evaluated from \( h(n) \) expression
\[
\begin{align*}
    h(1) &= h(-1) = 0 \\
    h(2) &= h(-2) = -0.3183 \\
    h(3) &= h(-3) = 0 \\
    h(4) &= h(-4) = 0 \\
    h(5) &= h(-5) = 0
\end{align*}
\]
• The transfer function of the filter is
\[
H(z) = h(0) + \sum_{n=1}^{(N-1)/2} [h(n)\{z^n + z^{-n}\}]
\]
\[
= 0.5 - 0.3183(z^2 + z^{-2})
\]
the transfer function of the realizable filter is
\[
H'(z) = z^{-5}[0.5 - 0.3183(z^2 + z^{-2})]
\]
\[
= -0.3183z^{-3} + 0.5z^{-5} - 0.3183z^{-7}
\]
the filter coeff are
\[
\begin{align*}
    h'(0) &= h'(10) = h'(1) = h'(9) = h'(2) = h'(8) = h'(4) = h'(6) = 0 \\
    h'(3) &= h'(7) = -0.3183 \\
    h'(5) &= 0.5
\end{align*}
\]
• The magnitude response can be expressed as
\[
|H(e^{j\omega})| = \sum_{n=1}^{(N-1)/2} a(n) \cos n\omega
\]
comparing this exp with
\[
|H(e^{j\omega})| = |z^{-5}[h(0) + 2\sum_{n=1}^{5} h(n) \cos n\omega]| 
\]
• We have
\[
\begin{align*}
    a(0) &= h(0) \\
    a(1) &= 2h(1) = 0 \\
    a(2) &= 2h(2) = -0.6366 \\
    a(3) &= 2h(3) = 0 \\
    a(4) &= 2h(4) = 0 \\
    a(5) &= 2h(5) = 0
\end{align*}
\]
The magnitude response function is
\[
|H(e^{j\omega})| = 0.5 - 0.6366 \cos 2\omega \text{ which can plotted for various values of } \omega \text{ in degrees } =\{0 20 30 45 60 75 90 105 120 135 150 160 180\};
\]
\[ |H(e^{j\omega})| \text{ in dBs} = [-17.3 -38.17 -14.8 -6.02 -1.74 0.4346 1.11 0.4346 -1.74 -6.02 -14.8 -38.17 -17.3]; \]

**Prob:** Design an ideal lowpass filter with a freq response

\[
H_d(e^{j\omega}) = 1 \quad \text{for} \quad -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2}
\]

\[= 0 \quad \text{for} \quad \frac{\pi}{2} \leq |\omega| \leq \pi
\]

Find the values of \( h(n) \) for \( N = 11 \). Find \( H(z) \). Plot the magnitude response

From the freq response we can determine \( h_d(n) \),

\[
h_d(n) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega = \frac{\sin \frac{\pi n}{2}}{\frac{\pi n}{2}} \quad -\infty \leq n \leq \infty \quad \text{and} \quad n \neq 0
\]

Truncating \( h_d(n) \) to 11 samples

\[
h(0) = 1/2
\]

\[
h(1) = h(-1) = 0.3183
\]

\[
h(2) = h(-2) = 0
\]

\[
h(3) = h(-3) = -0.106
\]

\[
h(4) = h(-4) = 0
\]

\[
h(5) = h(-5) = 0.06366
\]

The realizable filter can be obtained by shifting \( h(n) \) by 5 samples to right \( h'(n) = h(n-5) \)
Using the result of magnitude response for \( M \) odd and symmetry

\[
H_r(e^{i\omega}) = [h(M-1) + \sum_{n=0}^{M-3} h(n) \cos \omega(M-1-n)]
\]

\[
|H_r(e^{i\omega})| = |0.5 + 0.6366 \cos \omega - 0.212 \cos 3\omega + 0.127 \cos 5\omega|
\]

**Exercise Problem:**

Design an ideal band reject filter with a frequency response:

\[
H_d(e^{i\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \frac{\pi}{3} \text{ and } |\omega| \geq \frac{2\pi}{3} \\ 0 & \text{otherwise} \end{cases}
\]

Find the values of \( h(n) \) for \( M = 11 \) and plot the frequency response

\[
h(n) = [0 \, -0.1378 \, 0 \, 0.2757 \, 0 \, 0.667 \, 0 \, 0.2757 \, 0 \, -0.1378 \, 0];
\]

**Window based Linear Phase FIR filter design**

- The arbitrary truncation of impulse response obtained through inverse Fourier relation can lead to distortions in the final frequency response.
- The arbitrary truncation is equivalent to multiplying infinite length function with finite length rectangular window, i.e., \( h(n) = h_d(n) \cdot w(n) \) where \( w(n) = 1 \) for \( n = \pm(M-1)/2 \)
- The above multiplication in time domain corresponds to convolution in freq domain, i.e.,
  \[ H(e^{j\omega}) = H_d(e^{j\omega}) \ast W(e^{j\omega}) \]
  where \( W(e^{j\omega}) \) is the FT of window function \( w(n) \).
- The FT of \( w(n) \) is given by
  \[
  W(e^{j\omega}) = \frac{\sin(\omega M / 2)}{\sin(\omega / 2)}
  \]
• Suppose the filter to be designed is Low pass filter then the convolution of ideal filter freq response and window function freq response results in distortion in the resultant filter freq response. The ideal sharp cutoff chars are lost and presence of ringing effect is seen at the band edges which is referred to Gibbs Phenomena.
• This is due to main lobe width and side lobes of the window function freq response.
• The main lobe width introduces transition band and side lobes results in rippling characters in pass band and stop band.
• Smaller the main lobe width smaller will be the transition band
• The ripples will be of low amplitude if the peak of the first side lobe is far below the main lobe peak.

How to reduce the distortions?
• Increase length of the window
  - as M increases the main lob width becomes narrower, hence the transition band width is decreased
  - With increase in length the side lobe width is decreased but height of each side lobe increases in such a manner that the area under each sidelobe remains invariant to changes in M. Thus ripples and ringing effect in pass-band and stop-band are not changed.
• Choose windows which tapers off slowly rather than ending abruptly
  - Slow tapering reduces ringing and ripples but generally increases transition width since main lobe width of these kind of windows are larger.

What is ideal window characteristics?
• Window having very small main lobe width with most of the energy contained with it (i.e., ideal window freq response must be impulsive)
  ➢ Window design is a mathematical problem
  ➢ More complex the window lesser are the distortions
  ➢ Windows better than rectangular window are, Hamming, Hanning, Blackman, Bartlett, Traingular, Kaiser

Rectangular window

\[ w_r(n) = 1 \text{ for } 0 \leq n \leq M - 1 \]
Hanning windows:

\[ w_{\text{han}}(n) = 0.5(1 - \cos \frac{2\pi n}{M - 1}) \text{ for } 0 \leq n \leq M - 1 \]

Hamming windows:

\[ w_{\text{ham}}(n) = 0.54 - 0.46 \cos \frac{2\pi n}{M - 1} \text{ for } 0 \leq n \leq M - 1 \]

Blackman windows:

\[ w_{\text{blk}}(n) = 0.42 - 0.5 \cos \frac{2\pi n}{M - 1} + 0.08 \cos \frac{4\pi n}{M - 1} \text{ for } 0 \leq n \leq M - 1 \]
Bartlett (Triangular) windows:

\[ w_{\text{bart}}(n) = 1 - \frac{2 | n - \frac{M-1}{2} |}{M-1} \quad \text{for } 0 \leq n \leq M-1 \]

Kaiser windows:

\[ w_k(n) = \frac{I_0 \left( \frac{(M-1)^2}{2} - \left( n - \frac{M-1}{2} \right)^2 \right)}{I_0 \left( \alpha \left( \frac{M-1}{2} \right) \right)} \quad \text{for } 0 \leq n \leq M-1 \]
<table>
<thead>
<tr>
<th>Type of window</th>
<th>Appr. Transition width of the main lobe</th>
<th>Peak sidelobe (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>$4\pi/M$</td>
<td>-13</td>
</tr>
<tr>
<td>Bartlett</td>
<td>$8\pi/M$</td>
<td>-27</td>
</tr>
<tr>
<td>Hanning</td>
<td>$8\pi/M$</td>
<td>-32</td>
</tr>
<tr>
<td>Hamming</td>
<td>$8\pi/M$</td>
<td>-43</td>
</tr>
<tr>
<td>Blackman</td>
<td>$12\pi/M$</td>
<td>-58</td>
</tr>
</tbody>
</table>

**Procedure for designing linear-phase FIR filters using windows**

1. From the desired freq response using inverse FT relation obtain $h_d(n)$
2. Truncate the infinite length of the impulse response to finite length with (assuming $M$ odd) choosing proper window

   \[ h(n) = h_d(n)w(n) \text{ where } w(n) \text{ is the window function defined for } -(M-1)/2 \leq n \leq (M-1)/2 \]
3. Introduce $h(n) = h(-n)$ for linear phase characteristics
4. Write the expression for $H(z)$; this is non-causal realization
5. To obtain causal realization $H'(z) = z^{-(M-1)/2} H(z)$

**Prob:** Design an ideal highpass filter with a frequency response:

\[ H_d(e^{j\omega}) = 1 \quad \text{for } \frac{\pi}{4} \leq |\omega| \leq \pi \]

\[ = 0 \quad |\omega| < \frac{\pi}{4} \]

using a hanning window with $M = 11$ and plot the frequency response.
The Hamming window function is given by

\[
\begin{align*}
\text{w}_n(n) &= 0.5 + 0.5 \cos \left( \frac{2\pi n}{M - 1} \right) \\
&= 0 \quad \text{otherwise}
\end{align*}
\]

for \( N = 11 \)

\[
\begin{align*}
\text{w}_n(n) &= 0.5 + 0.5 \cos \left( \frac{\pi n}{5} \right) \\
&= 0 \quad \text{otherwise}
\end{align*}
\]
\[ h(n) = w_{hn}(n)h_d(n) \]
\[ h(n) = [0 \ 0 \ -0.026 \ -0.104 \ -0.204 \ 0.75 \ -0.204 \ -0.104 \ -0.026 \ 0 \ 0] \]
\[ h'(n) = h(n - 5) \]
\[ H'(z) = -0.026z^{-2} - 0.104z^{-3} - 0.204z^{-4} + 0.75z^{-5} - 0.204z^{-6} - 0.104z^{-7} - 0.026z^{-8} \]

Using the equation
\[ H_r(e^{j\omega}) = \left[ h\left(\frac{M-1}{2}\right) + 2 \sum_{n=0}^{M-3} h(n) \cos \left(\omega \frac{M-1}{2} - n\right) \right] \]
\[ H_r(e^{j\omega}) = 0.75 + 2 \sum_{n=0}^{4} h(n) \cos \omega(5 - n) \]

The magnitude response is given by,
\[ |H_r(e^{j\omega})| = |0.75 - 0.408 \cos \omega - 0.208 \cos 2\omega - 0.052 \cos 3\omega| \]
\[ \omega \text{ in degrees} = [0 \ 15 \ 30 \ 45 \ 60 \ 75 \ 90 \ 105 \ 120 \ 135 \ 150 \ 165 \ 180] \]
\[ |H(e^{j\omega})| \text{ in dBs} = [-21.72 \ -17.14 \ -10.67 \ -6.05 \ -3.07 \ -1.297 \ -0.3726 \ -0.0087 \ 0.052 \ 0.015 \ 0 \ 0 \ 0.017] \]

**Prob:** Design a filter with a frequency response:
\[ H_d(e^{j\omega}) = e^{-j3\omega} \quad \text{for} \quad -\frac{\pi}{4} \leq \omega \leq \frac{\pi}{4} \]
\[ = 0 \quad \frac{\pi}{4} < |\omega| \leq \pi \]

using a Hanning window with \( M = 7 \)
Soln:
The freq resp is having a term $e^{-j(M-1)/2}$ which gives $h(n)$ symmetrical about $n = M-1/2 = 3$ i.e we get a causal sequence.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j3\omega} e^{j\omega n} d\omega$$

$$\sin \frac{\pi}{4} (n-3) = \frac{\sin \pi(n-3)}{\pi(n-3)}$$

this gives $h_d(0) = h_d(6) = 0.075$
$h_d(1) = h_d(5) = 0.159$
$h_d(2) = h_d(4) = 0.22$
$h_d(3) = 0.25$

The Hanning window function values are given by
$w_{hn}(0) = w_{hn}(6) = 0$
$w_{hn}(1) = w_{hn}(5) = 0.25$
$w_{hn}(2) = w_{hn}(4) = 0.75$
$w_{hn}(3) = 1$

$h(n) = h_d(n) w_{hn}(n)$
h(n) = [0 0.03975 0.165 0.25 0.165 0.3975 0]

Design of Linear Phase FIR filters using Frequency Sampling method

Motivation: We know that DFT of a finite duration DT sequence is obtained by sampling FT of the sequence then DFT samples can be used in reconstructing original time
domain samples if frequency domain sampling was done correctly. The samples of FT of \(h(n)\) i.e., \(H(k)\) are sufficient to recover \(h(n)\).

Since the designed filter has to be realizable then \(h(n)\) has to be real, hence even symmetry properties for mag response \(|H(k)|\) and odd symmetry properties for phase response can be applied. Also, symmetry for \(h(n)\) is applied to obtain linear phase chas.

For DFT relationship we have

\[
h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k)e^{j\frac{2\pi k n}{N}} \quad \text{for} \quad n = 0,1,\ldots,N-1
\]

\[
H(k) = \sum_{n=0}^{N-1} h(n)e^{-j\frac{2\pi k n}{N}} \quad \text{for} \quad k = 0,1,\ldots,N-1
\]

Also we know \(H(k) = H(z)|_{z=e^{j2\pi k/N}}\)

The system function \(H(z)\) is given by

\[
H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}
\]

Substituting for \(h(n)\) from IDFT relationship

\[
H(z) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) \sum_{n=0}^{N-1} \frac{1}{1-e^{j2\pi k n/N}} z^{-1}
\]

Since \(H(k)\) is obtained by sampling \(H(e^{j\omega})\) hence the method is called **Frequency Sampling Technique**.

Since the impulse response samples or coefficients of the filter has to be real for filter to be realizable with simple arithmetic operations, properties of DFT of real sequence can be used. The following properties of DFT for real sequences are useful:

\[H^*(k) = H(N-k)\]

\[|H(k)| = |H(N-k)|\] - magnitude response is even

\[\theta(k) = -\theta(N-k)\] – Phase response is odd

\[
h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k)e^{j\frac{2\pi k n}{N}} \quad \text{can be rewritten as (for N odd)}
\]
\[
\begin{align*}
  h(n) &= \frac{1}{N} \left[ H(0) + \sum_{k=1}^{N-1} H(k) e^{j2\pi kn/N} \right] \\
  h(n) &= \frac{1}{N} \left[ H(0) + \sum_{k=1}^{N-1/2} H(k) e^{j2\pi kn/N} + \sum_{k=N-1/2}^{N-1} H(k) e^{-j2\pi kn/N} \right]
\end{align*}
\]

Using substitution \( k = N - r \) or \( r = N - k \) in the second substitution with \( r \) going from now \( (N-1)/2 \) to 1 as \( k \) goes from 1 to \((N-1)/2\)

\[
\begin{align*}
  h(n) &= \frac{1}{N} \left[ H(0) + \sum_{k=1}^{(N-1)/2} H(k) e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} H(N-k) e^{-j2\pi kn/N} \right] \\
  h(n) &= \frac{1}{N} \left[ H(0) + \sum_{k=1}^{(N-1)/2} H(k) e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} H^*(k) e^{-j2\pi kn/N} \right] \\
  h(n) &= \frac{1}{N} \left[ H(0) + \sum_{k=1}^{(N-1)/2} H(k) e^{j2\pi kn/N} + \sum_{k=1}^{(N-1)/2} (H(k) e^{j2\pi kn/N})^* \right] \\
  h(n) &= \frac{1}{N} \left[ H(0) + 2 \sum_{k=1}^{(N-1)/2} \Re(H(k) e^{j2\pi kn/N}) \right]
\end{align*}
\]

Similarly for \( N \) even we have

\[
\begin{align*}
  h(n) &= \frac{1}{N} \left[ H(0) + 2 \sum_{k=1}^{(N-1)/2} \Re(H(k) e^{j2\pi kn/N}) \right]
\end{align*}
\]

Using the symmetry property \( h(n) = h(N-1-n) \) we can obtain Linear phase FIR filters using the frequency sampling technique.

**Prob:** Design a LP FIR filter using Freq sampling technique having cutoff freq of \( \pi/2 \) rad/sample. The filter should have linear phase and length of 17.

The desired response can be expressed as

\[
\begin{align*}
  H_d(e^{j\omega}) &= e^{-j\omega(M-1)/2} \quad for \quad |\omega| \leq \omega_c \\
  &= 0 \quad otherwise \\
  \text{with} \quad M = 17 \quad \text{and} \quad \omega_c = \pi/2
\end{align*}
\]
\[ H_d(e^{j\omega}) = e^{-j\omega \theta} \quad \text{for} \quad 0 \leq \omega \leq \pi / 2 \]
\[ = 0 \quad \text{for} \quad \pi / 2 \leq \omega \leq \pi \]

Selecting \( \omega_k = \frac{2\pi k}{M} = \frac{2\pi k}{17} \) \( \text{for} \quad k = 0,1,\ldots,16 \)

\[ H(k) = H_d(e^{j\omega}) \bigg|_{\omega = \frac{2\pi k}{17}} \]

\[ H(k) = e^{-\frac{2\pi k}{17}} \quad \text{for} \quad 0 \leq \frac{2\pi k}{17} \leq \frac{\pi}{2} \]
\[ = 0 \quad \text{for} \quad \pi / 2 \leq \frac{2\pi k}{17} \leq \pi \]
\[ H(k) = e^{-\frac{16\pi k}{17}} \quad \text{for} \quad 0 \leq k \leq \frac{17}{4} \]
\[ = 0 \quad \text{for} \quad \frac{17}{4} \leq k \leq \frac{17}{2} \]

The range for “k” can be adjusted to be an integer such as

\[ 0 \leq k \leq 4 \]
and \( 5 \leq k \leq 8 \)

The freq response is given by

\[ H(k) = e^{-\frac{2\pi k}{17}} \quad \text{for} \quad 0 \leq k \leq 4 \]
\[ = 0 \quad \text{for} \quad 5 \leq k \leq 8 \]

Using these value of \( H(k) \) we obtain \( h(n) \) from the equation

\[ h(n) = \frac{1}{M} (H(0) + 2 \sum_{k=1}^{(M-1)/2} \text{Re}(H(k)e^{j2\pi kn/M})) \]

i.e.,

\[ h(n) = \frac{1}{17} (1 + 2 \sum_{k=3}^{4} \text{Re}(e^{-j16\pi k/17} e^{j2\pi kn/17})) \]
\[ h(n) = \frac{1}{17} (H(0) + 2 \sum_{k=1}^{4} \cos\left(\frac{2\pi k(8-n)}{17}\right)) \quad \text{for} \quad n = 0,1,\ldots,16 \]
• Even though \( k \) varies from 0 to 16 since we considered \( \omega \) varying between 0 and \( \pi/2 \) only \( k \) values from 0 to 8 are considered
• While finding \( h(n) \) we observe symmetry in \( h(n) \) such that \( n \) varying 0 to 7 and 9 to 16 have same set of \( h(n) \)

**Design of FIR Differentiator**

• Differentiator are widely used in Digital and Analog systems whenever a derivative of the signal is needed
• Ideal differentiator has pure linear magnitude response in the freq range \(-\pi\) to \(+\pi\)

![Diagram](attachment:image.png)

**Prob:** Design an Ideal Differentiator using a) rectangular window and b) Hamming window with length of the system = 7.

Solution:
As seen from differentiator frequency chars. It is defined as

\[
H(e^{j\omega}) = j\omega \quad \text{between } -\pi \text{ to } +\pi
\]

\[
h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega = \frac{\cos \frac{\pi n}{\pi}}{n} \quad -\infty \leq n \leq \infty \quad \text{and} \quad n \neq 0
\]

The \( h_d(n) \) is an add function with \( h_d(n) = -h_d(-n) \) and \( h_d(0) = 0 \)

a) rectangular window

\[h(n) = h_d(n)w_r(n)\]

\[h(1) = -h(-1) = h_d(1) = -1\]
h(2) = -h(-2) = h_d(2) = 0.5
h(3) = -h(-3) = h_d(3) = -0.33

h'(n) = h(n-3) for causal system
thus,

\[ H'(z) = 0.33 - 0.5 z^{-1} + z^{-2} - z^{-4} + 0.5 z^{-5} - 0.33 z^{-6} \]

Also from the equation

\[ H_r(e^{j\omega}) = 2 \sum_{n=0}^{(M-3)/2} h(n) \sin \frac{\omega (M-1)}{2} - n \]

For M=7 and h'(n) as found above we obtain this as

\[ H_r(e^{j\omega}) = 0.66 \sin 3\omega - \sin 2\omega + 2 \sin \omega \]

\[ H(e^{j\omega}) = jH_r(e^{j\omega}) = j(0.66 \sin 3\omega - \sin 2\omega + 2 \sin \omega) \]

b) Hamming window

h(n) = h_d(n) w_h(n)

where \( w_h(n) \) is given by

\[ w_h(n) = 0.54 + 0.46 \cos \frac{2\pi n}{(M-1)} \quad -(M-1)/2 \leq n \leq (M-1)/2 \]

\[ = 0 \quad otherwise \]

For the present problem

\[ w_h(n) = 0.54 + 0.46 \cos \frac{\pi n}{3} \quad -3 \leq n \leq 3 \]

The window function coefficients are given by for n=-3 to +3

\[ \text{Wh(n)} = [0.08 0.31 0.77 1 0.77 0.31 0.08] \]

Thus \( h'(n) = h(n-5) = [0.0267, -0.155, 0.77, 0, -0.77, 0.155, -0.0267] \)

Similar to the earlier case of rectangular window we can write the freq response of differentiator as

\[ H(e^{j\omega}) = jH_r(e^{j\omega}) = j(0.0534 \sin 3\omega - 0.31 \sin 2\omega + 1.54 \sin \omega) \]
We observe
- With rectangular window, the effect of ripple is more and transition band width is small compared with hamming window
- With hamming window, effect of ripple is less whereas transition band is more

**Design of FIR Hilbert transformer**
- Hilbert transformers are used to obtain phase shift of 90 degree
- They are also called j operators
- These signals are needed in quadrature signal processing
- The Hilbert transformer is very useful when out of phase component (or imaginary part) need to be generated from available real component of the signal.

Design an ideal Hilbert transformer using a) rectangular window and b) Blackman Window with M = 11
Solution:
As seen from freq chars it is defined as

\[ H_d(e^{j\omega}) = j \quad -\pi \leq \omega \leq 0 \]
\[ = -j \quad 0 \leq \omega \leq \pi \]

The impulse response is given by

\[
h_d(n) = \frac{1}{2\pi} \left[ \int_{-\pi}^{0} e^{j\omega n} d\omega + \int_{0}^{\pi} e^{j\omega n} d\omega \right] = \frac{(1 - \cos \frac{\pi n}{m})}{m} \quad -\infty \leq n \leq \infty \quad \text{except} \quad n = 0
\]

At \( n = 0 \) it is \( h_d(0) = 0 \) and \( h_d(n) \) is an odd function

a) Rectangular window
\[ h(n) = h_d(n) \quad w_r(n) = h_d(n) \quad \text{for} \quad -5 \geq n \geq 5 \]
\[ h'(n) = h(n-5) \]
\[ h(n) = [-0.127, 0, -0.212, 0, -0.636, 0, 0.636, 0, 0.212, 0, 0.127] \]
\[ H_r(e^{j\omega}) = 2 \sum_{n=0}^{4} h(n) \sin \omega(5 - n) \]
\[ H(e^{j\omega}) = j | H_r(e^{j\omega}) | = j[0.254 \sin 5\omega + 0.424 \sin 3\omega + 1.272 \sin \omega] \]

b) Blackman Window
window function is defined as
\[ w_b(n) = 0.42 + 0.5 \cos \frac{\pi n}{5} + 0.08 \cos \frac{2\pi n}{5} \quad -5 \leq n \leq 5 \]
\[ = 0 \quad \text{otherwise} \]
\[ W_b(n) = [0, 0.04, 0.2, 0.509, 0.849, 1.0, 0.849, 0.509, 0.2, 0.04, 0] \quad \text{for} \quad -5 \geq n \geq 5 \]
\[ h'(n) = h(n-5) = [0, 0, -0.0424, 0, -0.5405, 0, 0.5405, 0, 0.0424, 0, 0] \]
\[ H(e^{j\omega}) = -j[0.0848 \sin 3\omega + 1.0810 \sin \omega] \]
**Frequency Transformation**

**Why Frequency Transformation:**

- This allows one to design prototype filter and transform to any specific frequency selective type
- Designers can concentrate on improved methods of designing prototype rather than wasting time on devising design methodologies for different types of filters
- One design and all types of frequency selective filters is an advantage

**Where Frequency Transformation**

- **Analog Domain**
- **Digital Domain**

**Transformation in Digital Domain**

Frequency Transformation in Analog domain:

- Normalized Low Pass filter with cutoff freq of $\Omega_p = 1 \text{ rad/sec}$ is designed

Normalized LPF to LPF of specific cutoff:

$$s \rightarrow \frac{s}{\Omega_p}$$

$$H_1(s) = H_p \left( \frac{\Omega_p}{\Omega_p} s \right)$$

Where, 

$\Omega_p$ = normalized cutoff freq=1 rad/sec 
$\Omega_p'$ = Desired LP cutoff freq

at $\Omega = \Omega_p$ it is $H(j1)$
Frequency Transformation in Digital Domain

- Involves replacing the variable $Z^{-1}$ by a rational function $g(z^{-1})$
- Should satisfy the following properties.
  
  ➢ Mapping $Z^{-1}$ to $g(z^{-1})$ must map points inside the unit circle in the $Z$-plane onto the unit circle of $z$-plane.
  
  ➢ For stable filter, the inside of the unit circle of the $Z$-plane must map onto the inside of the unit circle of the $z$-plane.

The general form of the function $g(.)$ that satisfy the above requirements of "all-pass" type is

$$g(z^{-1}) = \pm \prod_{k=1}^{\infty} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}$$

<table>
<thead>
<tr>
<th>Type of Transformation</th>
<th>Transformation</th>
<th>Band edge frequencies of new filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>$S \rightarrow \frac{\Omega_0}{\Omega_p} S$</td>
<td>$\Omega_p'$</td>
</tr>
<tr>
<td>HP</td>
<td>$S \rightarrow \frac{\Omega_p \Omega_0}{S}$</td>
<td>$\Omega_p'$</td>
</tr>
<tr>
<td>BP</td>
<td>$S \rightarrow \Omega_p \frac{s^2 + \Omega_1 \Omega_u}{s(\Omega_1 - \Omega_u)}$</td>
<td>$\Omega_1, \Omega_u$</td>
</tr>
<tr>
<td>BS</td>
<td>$S \rightarrow \Omega_p \frac{s(\Omega_u - \Omega_c)}{s^2 + \Omega_u \Omega_1}$</td>
<td>$\Omega_1, \Omega_u$</td>
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<th>Type of Transformation</th>
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</tr>
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<tr>
<td>Lowpass</td>
<td>$z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$</td>
<td>$\omega_c=$cutoff frequency of new filter [\alpha = \frac{\sin((\omega'_c - \omega_c)/2)}{\sin((\omega'_c + \omega_c)/2)}]</td>
</tr>
<tr>
<td>Highpass</td>
<td>$z^{-1} \rightarrow -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$</td>
<td>$\omega_c=$cutoff frequency of new filter [\alpha = -\frac{\cos((\omega'_c + \omega_c)/2)}{\cos((\omega'_c - \omega_c)/2)}]</td>
</tr>
</tbody>
</table>
Prob:
Let
\[ H(s) = \frac{1}{s^2 + s + 1} \]

Represents the transfer function of a lowpass filter (not butterworth) with a passband of 1 rad/sec. Use freq transformation to find the transfer function of the following filters:
1. A LP filter with a passband of 10 rad/sec
2. A HP filter with a cutoff freq of 1 rad/sec
3. A HP filter with a cutoff freq of 10 rad/sec
4. A BP filter with a passband of 10 rad/sec and a corner freq of 100 rad/sec
5. A BS filter with a stopband of 2 rad/sec and a center freq of 10 rad/sec

Solution:
Given
\[ H(s) = \frac{1}{s^2 + s + 1} \]

a. LP – LP Transform
replace
\[ s \rightarrow \frac{s}{\Omega_p} = \frac{s}{10} \]

\[ H_a(s) = H(s) \bigg|_{s \rightarrow \frac{s}{10}} = \frac{1}{\left(\frac{s}{10}\right)^2 + \left(\frac{s}{10}\right) + 1} \]

\[ = \frac{100}{s^2 + 10s + 100} \]
b. LP – HP(normalized) Transform

\[ s \rightarrow \frac{\Omega_u}{s} = \frac{1}{s} \]

\[ sub \quad H_o(s) = H(s) \bigg|_{s \rightarrow \frac{1}{s}} = \frac{1}{\left(\frac{1}{s}\right)^2 + \left(\frac{1}{s}\right) + 1} \]

\[ = \frac{s^2}{s^2 + s + 1} \]

c. LP – HP(specified cutoff) Transform

replace

\[ s \rightarrow \frac{\Omega_u}{s} = \frac{10}{s} \]

\[ sub \quad H_o(s) = H(s) \bigg|_{s \rightarrow \frac{10}{s}} = \frac{1}{\left(\frac{10}{s}\right)^2 + (\frac{10}{s}) + 1} \]

\[ = \frac{s^2}{s^2 + 10s + 100} \]

c. LP – BP Transform

replace

\[ s \rightarrow \frac{s^2 + \Omega_u^{2} - \Omega_l^{2}}{s(\Omega_u - \Omega_l)} = \frac{s^2 + \Omega_u^{2}}{sB_0} \quad where \quad \Omega_o = \sqrt{\Omega_u \Omega_l} \]

and \( B_o = (\Omega_u - \Omega_l) \)

\[ sub \quad H_o(s) = H(s) \bigg|_{s \rightarrow \frac{s^2 + 10^4}{100s}} \]

\[ = \frac{100s^2}{s^4 + 10s^3 + 20100s^2 + 10^5s + 10^8} \]

c. LP – BS Transform

replace

\[ s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u^{2} - \Omega_l^{2}} = \frac{sB_0}{s^2 + \Omega_o^2} \quad where \quad \Omega_o = \sqrt{\Omega_u \Omega_l} \]

and \( B_o = (\Omega_u - \Omega_l) \)

\[ sub \quad H_o(s) = H(s) \bigg|_{s \rightarrow \frac{2s}{s^2 + 100}} \]

\[ = \frac{(s^2 + 100)^2}{s^4 + 2s^3 + 204s^2 + 200s + 10^4} \]
**Prob:**

Convert single pole LP Butterworth filter with system function

\[ H(z) = \frac{0.245(1 + z^{-1})}{1 + 0.509z^{-1}} \]

Into BPF with upper & lower cutoff frequency \( \omega_u \) & \( \omega_l \) respectively.

The LPF has 3-dB bandwidth \( \omega_p = 0.2\pi \)

- **Soln**

\[ z^{-1} \rightarrow \frac{z^{-2} - \alpha_1z^{-1} + \alpha_2}{\alpha_2z^{-2} - \alpha_1z^{-1} + 1} \]

\[ H(Z) = \frac{0.245(1 + \frac{z^{-2} - \alpha_1z^{-1} + \alpha_2}{\alpha_2z^{-2} - \alpha_1z^{-1} + 1})}{1 + 0.509} \left( \frac{z^{-2} - \alpha_1z^{-1} + \alpha_2}{\alpha_2z^{-2} - \alpha_1z^{-1} + 1} \right) \]

\[ H[z] = \frac{0.245(1 - \alpha_2)(1 - z^{-2})}{(1 + 0.509\alpha_2) - 1.509\alpha_1z^{-1} + (\alpha_2 + 0.509)z^{-2}} \]

Note that the resulting filter has zeros at \( z=\pm1 \) and a pair of poles that depend on the choice of \( \omega_l \) and \( \omega_u \)

**Ex:**

\[ \omega_u = \frac{3\pi}{5}, \quad \omega_l = \frac{2\pi}{5} \]

\[ \omega_p = 0.2\pi \]

Then \( k=1, \quad \alpha_2 = 0, \alpha_1 = 0 \)

\[ \therefore H[z] = \frac{0.245(1 - z^{-2})}{1 + 0.509z^{-2}} \]

This filter has poles at \( z=\pm j0.713 \) and hence resonates at \( \omega=\pi/2 \)

- **Conclusion**
  - It is shown here that how easy to convert one form of filter design to another form.
  - What we require is only prototype low pass filter design steps to transform to any other form.