Signals and Systems

Course material

VTU Edusat program

By

Dr. Uma Mudenagudi

uma@bvb.edu

Department of Electronics and Communication

BVCET, Hubli-580030

May 22, 2009

Contents

List of Figures

Chapter 1

Introduction

This course material on signals and systmes is a part of VTU-edusat program. The course has been carried out by three faculty from different engineering colleges across the state. We divided the course into three parts as given:

- Part 1 by Dr.B.Kanmani, BMSCE, Bangalorelore (Coordinator)
  - Unit I: Introduction
  - Units II and III: Time domain Analysis (partial)
- Part 2 by Dr.R.Krupa, KLE, Belgaum
  - Units IV, V, VI: Fourier representation
- Part 3 by Dr. Uma Mudenagudi, BVBCET, Hubli
  - Units II and III: Time domain Analysis (partial)
  - Units VII, VIII: Z-domain Analysis

In what follows, I give the summary of the 11 classes carried during 11 April 2009 to 12 May 2009. The text book followed is Signals and Systems by Simon Haykin and Barry Van Veen [2]. The other reference books used for the class include [4, 1, 3, 5]. Each section describes one class.

1.1 Class 1: Difference and differential equation

Outline of today's class

- Brief review
- Difference equation
- Differential equation
- Solution to difference and differential equation
- Homogeneous and particular solution

1.1.1 Brief review

- Signals
  - classification, operations elementary signals
  - Fourier representations of signals and and applications
- Systems
  - properties
  - time domain representations, convolution, properties of impulse response

Time domain representation of LTI Systems

- Impulse response: characterizes the behavior of any LTI system
- Linear constant coefficient differential or difference equation: input output behavior
- Block diagram: as an interconnection of three elementary operations

1.1.2 Differential and difference equation

- General form of differential equation is

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \tag{1.1}$$

- General form of difference equation is

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{1.2}$$

- where a_k and b_k are coefficients, $x(.)$ is input and $y(.)$ is output and order of differential or difference equation is (M,N)

Example of Differential equation

- Consider the RLC circuit as shown in Figure 1.1. Let $x(t)$ be the input voltage source and $y(t)$ be the output current. Then summing up the voltage drops around the loop gives

$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

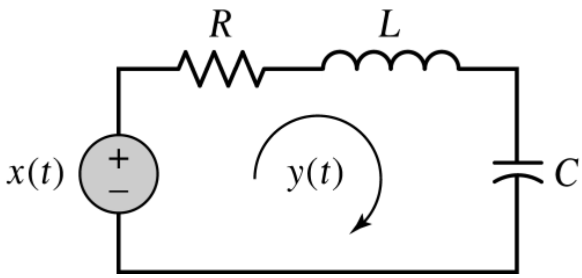


Figure 1.1: RLC circuit

Difference equation

- A wide variety of discrete-time systems are described by linear difference equations:

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \qquad n = 0, 1, 2, \ldots$$

  where the coefficients $a_1, \ldots, a_N$ and $b_0, \ldots, b_M$ do not depend on n. In order to be able to compute the system output, we also need to specify the initial conditions (ICs) $y[-1], y[-2] \ldots y[-N]$

- Systems of this kind are
  - linear time-invariant (LTI): easy to verify by inspection
  - causal: the output at time n depends only on past outputs $y[n-1], \ldots, y[n-N]$ and on current and past inputs $x[n], x[n-1], \ldots, x[n-M]$

- Systems of this kind are also called Auto Regressive Moving-Average (ARMA) filters. The name comes from considering two special cases.

- auto regressive (AR) filter of order N, $AR(N)$: $b_0 = \ldots = b_M = 0$

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = 0 \qquad n = 0, 1, 2, \ldots$$

In the AR case, the system output at time n is a linear combination of N past outputs; need to specify the ICs $y[-1], \ldots, y[-N]$.

- moving-average (MA) filter of order $N, AR(N) : a_0 = \ldots = a_N = 0$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad n = 0, 1, 2, \ldots$$

In the MA case, the system output at time n is a linear combination of the current input and M past inputs; no need to specify ICs.

- An ARMA(N, M) filter is a combination of both.
- Let us first rearrange the system equation:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \qquad n = 0, 1, 2, \ldots$$

- at $n = 0$

$$y[0] = \underbrace{-\sum_{k=1}^{N} a_k y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[-k]}_{depends\ on\ input\ x[0] \rightarrow x[-M]}$$

- at $n = 1$

$$y[1] = \sum_{k=1}^{N} a_k y[1-k] + \sum_{k=0}^{M} b_k x[1-k]$$

After rearranging

$$y[1] = -a_1 y[0] - \underbrace{\sum_{k=1}^{N-1} a_{k+1} y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[1-k]}_{depends\ on\ input x[1]...x[1-M]}$$

- at $n = 2$

$$y[2] = \sum_{k=1}^{N} a_k y[2-k] + \sum_{k=0}^{M} b_k x[2-k]$$

After rearranging

$$y[2] = -a_1 y[1] - a_2 y[0] - \underbrace{\sum_{k=1}^{N-1} a_{k+1} y[-k]}_{depends\ on\ ICs} + \underbrace{\sum_{k=0}^{M} b_k x[1-k]}_{depends\ on\ input\ x[2]...x[2-M]}$$

Implementation complexity

- In general, to compute the output of an $ARMA(N,M)$ filter at time n, we need the outputs at times $n-1, n-2, \ldots, n-N$ and the inputs at times $n, n-1, \ldots, n-M$
- memory: at any time, need to store N output values and $M+1$ input values, for a total of $N+M+1$ values
- operations: at any time n, wee need $N+M$ additions and $N+M+1$ multiplications, for a total of $2(N+M)+1$ operations to compute $y[n]$
- Computational complexity is proportional to $L = N+M$ and is independent of n

Example of Differential equation

- For a series RLC circuit with voltage source $x(t)$ and output current around the loop, $y(t)$ is

$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

- Differentiating with t gives

$$\frac{1}{C}y(t) + R\frac{d}{dt}y(t) + L\frac{d^2}{dt^2}y(t) = \frac{d}{dt}x(t)$$

- This is second order differential equation indicating two energy storage devices (cap and inductor)

Example of Difference equation

- An example of II order difference equation is

$$y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$$

- Memory in discrete system is analogous to energy storage in continuous system
- Number of initial conditions required to determine output is equal to maximum memory of the system

Initial conditions

- Initial conditions summarize all the information about the systems past that is needed to determine the future outputs

- In discrete case, for an N^{th} order system the N initial value are

$$y[-N], y[-N+1], \ldots, y[-1]$$

- The initial conditions for N^{th}-order differential equation are the values of the first N derivatives of the output

$$y(t)|_{t=0}, \frac{d}{dt}y(t)|_{t=0}, \frac{d^2}{dt^2}y(t)|_{t=0}, \ldots \frac{d^{N-1}}{dt^{N-1}}y(t)|_{t=0}$$

Solving difference equation

- Consider an example of difference equation $y[n] + ay[n-1] = x[n], \quad n = 0, 1, 2 \ldots$ with $y[-1] = 0$ Then

$$\begin{aligned}
y[0] &= -ay[-1] + x[0] \\
y[1] &= -ay[0] + x[1] \\
&= -a(-ay[-1] + x[0]) + x[1] \\
&= a^2y[-1] - ax[0]) + x[1] \\
y[2] &= -ay[1] + x[2] \\
&= -a(-a^2y[-1] - ax[0] + x[1]) + x[2] \\
&= a^3y[-1] + a^2x[0] - ax[1] + x[2]
\end{aligned}$$

and so on

- We get $y[n]$ as a sum of two terms:
$y[n] = (-a)^{n+1}y[-1] + \sum_{i=0}^{n}(-a)^{n-i}x[i], \qquad n = 0, 1, 2, \ldots$
- First term $(-a)^{n+1}y[-1]$ depends on IC's but not on input

- Second term $\sum_{i=0}^{n}(-a)^{n-i}x[i]$ depends only on the input, but not on the IC's
- This is true for any ARMA (auto regressive moving average) system: the system output at time n is a sum of the AR-only and the MA-only outputs at time n.
- Consider an ARMA (N,M) system $y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i], \quad n = 0,1,2,\ldots$ with the initial conditions $y[-1],\ldots,y[-N]$.
- Output at time n is:
$$y[n] = y_h[n] + y_p[n]$$
where $y_h[n]$ and $y_p[n]$ are homogeneous and particular solutions
- First term depends on IC's but not on input
- Second term depends only on the input, but not on the IC's
- Note that $y_h[n]$ is the output of the system determined by the ICs only (setting the input to zero), while $y_p[n]$ is the output of the system determined by the input only (setting the ICs to zero).
- $y_h[n]$ is often called the zero-input response (ZIR) usually referred as homogeneous solution of the filter (referring to the fact that it is determined by the ICs only)
- $y_p[n]$ is called the zero-state response (ZSR) usually referred as particular solution of the filter (referring to the fact that it is determined by the input only, with the ICs set to zero).

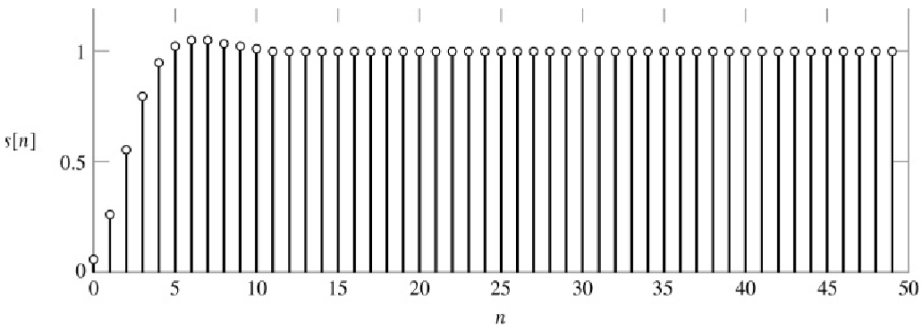


Figure 1.2: Step response

- Consider the output decomposition $y[n] = y_h[n] + y_p[n]$ of an ARMA (N, M) filter

$$y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i], \qquad n = 0, 1, 2, \ldots$$

  with the ICs $y[-1], \ldots, y[-N]$.

- The output of an ARMA filter at time n is the sum of the ZIR and the ZSR at time n.

Example of difference equation

- example: A system is described by $y[n] - 1.143y[n-1] + 0.4128y[n-2] = 0.0675x[n] + 0.1349x[n-1] + 0.675x[n-2]$

- Rewrite the equation as $y[n] = 1.143y[n-1] - 0.4128y[n-2] + 0.0675x[n] + 0.1349x[n-1] + 0.675x[n-2]$

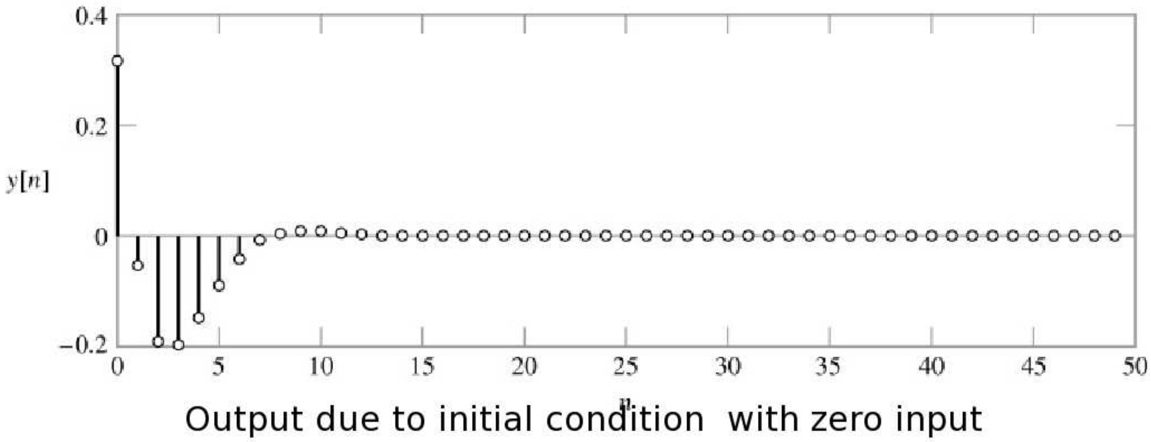


Figure 1.3: Initial condition

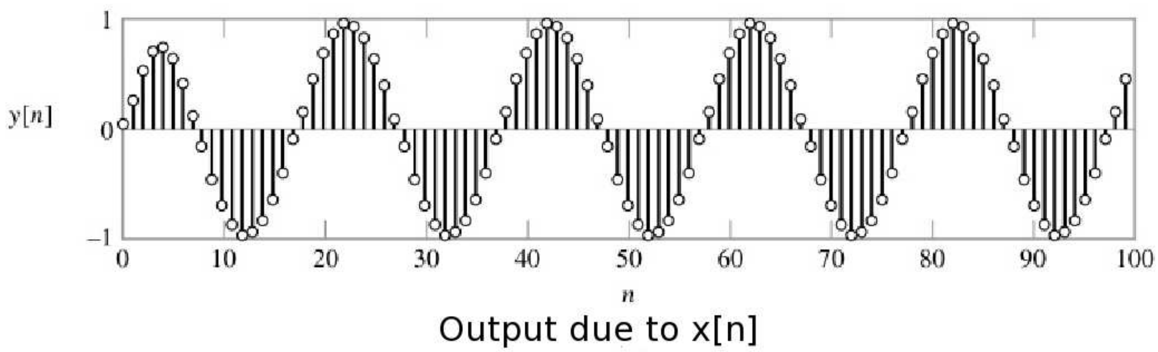


Figure 1.4: Due to input $x[n] = \cos(\frac{1}{10}\pi n)$

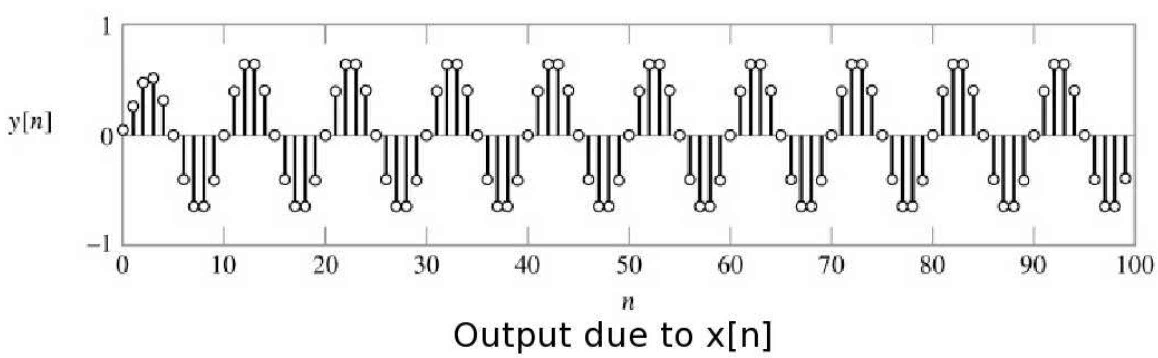


Figure 1.5: Due to input $x[n] = cos(\frac{1}{5}\pi n)$

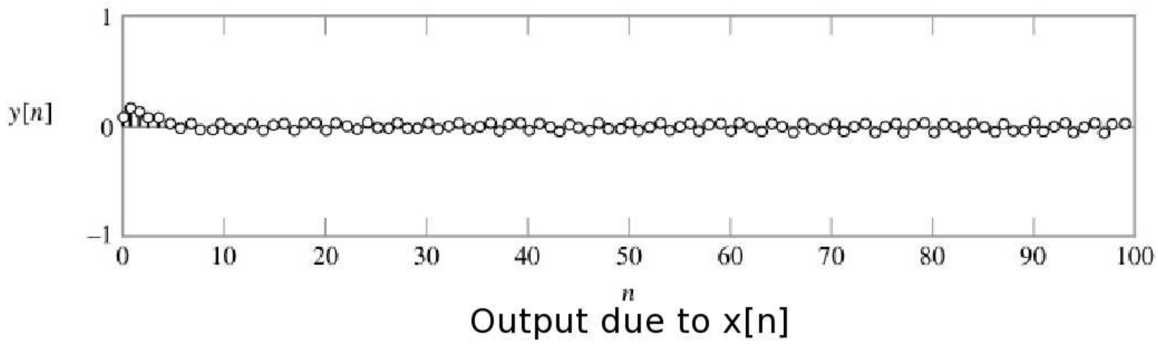


Figure 1.6: Due to input $x[n] = cos(\frac{7}{10}\pi n)$

Solving differential equation

- We will switch to continuous-time systems. A wide variety of continuous-time systems are described the linear differential equations:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \tag{1.3}$$

- Just as before, in order to solve the equation for $y(t)$, we need the ICs. In this case, the ICs are given by specifying the value of y and its derivatives 1 through $N-1$ at $t = 0^-$ (time "just before" $t = 0$): $y(0^-), y^{(1)}(0^-), \ldots, y^{(N-1)}(0^-)$. where $y^{(i)}(t)$ denotes the i^{th} derivative of $y(t)$, and $y^{(0)}(t) = y(t)$.
- Note: the ICs are given at $t = 0^-$ to allow for impulses and other discontinuities at $t = 0$.
- Systems described in this way are
- linear time-invariant (LTI): easy to verify by inspection

- causal: the value of the output at time t depends only on the output and the input at times $0 \leq \tau \leq t$
- As in the case of discrete-time system, the solution $y(t)$ can be decomposed into $y(t) = y_h(t) + y_p(t)$
  where homogeneous solution or zero-input response (ZIR), $y_h(t)$ satisfies the equation

$$y_h^N(t) + \sum_{i=0}^{N-1} a_i y_h^{(i)}(t) = 0, \qquad t \geq 0$$

  with the ICs $y^{(1)}(0^-), \ldots,$

- The zero-state response (ZSR) or particular solution $y_p(t)$ satisfies the equation

$$y_h^N(t) + \sum_{i=0}^{N-1} a_i y_h^{(i)}(t) = \sum_{i=0}^{m} b_i x^{(M-i)}(t), \qquad t \geq 0$$

  with ICs $y_p(0^-) = y_p^{(1)}(0^-) = ... = y_p^{(N-1)}(0^-) = 0.$

Homogeneous solution (ZIR) for CT

- A standard method for obtaining the homogeneous solution or (ZIR) is by setting all terms involving the input to zero.

$$\sum_{i=0}^{N} a_i y_h^{(i)}(t) = 0, \qquad t \geq 0$$

and homogeneous solution is of the form

$$y_h(t) = \sum_{i=1}^{N} C_i e^{r_i t}$$

where r_i are the N roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r^k = 0$$

and $C_1, \ldots, C_N$ are solved using ICs.

Homogeneous solution (ZIR) for DT

- The solution of the homogeneous equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

is

$$y_h[n] = \sum_{i=1}^{N} c_i r_i^n$$

where r_i are the N roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r^{N-k} = 0$$

and $C_1, \ldots, C_N$ are solved using ICs.

Example 1 (ZIR)

- Solution of

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 2x(t) + \frac{d}{dt}x(t)$$

is

$$y_h(t) = c_1 e^{-3t} + c_2 e^{-2t}$$

- Solution of $y[n] - 9/16y[n-2] = x[n-1]$ is $y_h[n] = c_1(3/4)^n + c_2(-3/4)^n$

Example 2 (ZIR)

- Consider the first order recursive system described by the difference equation $y[n] - \rho y[n-1] = x[n]$, find the homogeneous solution.
- The homogeneous equation (by setting input to zero) is $y[n] - \rho y[n-1] = 0$.
- The homogeneous solution for $N = 1$ is $y_h[n] = c_1 r_1^n$.
- r_1 is obtained from the characteristics equation $r_1 - \rho = 0$, hence $r_1 = \rho$
- The homogeneous solution is $y_h[n] = c_1 \rho^n$

Example 3 (ZIR)

- Consider the RC circuit described by $y(t) + RC\frac{d}{dt}y(t) = x(t)$
- The homogeneous equation is $y(t) + RC\frac{d}{dt}y(t) = 0$
- Then the homogeneous solution is

  $$y_h(t) = c_1 e^{r_1 t}$$

  where r_1 is the root of characteristic equation $1 + RCr_1 = 0$
- This gives $r_1 = -\frac{1}{RC}$

- The homogeneous solution is

$$y_h(t) = c_1 e^{\frac{-t}{RC}}$$

Particular solution (ZSR)

- Particular solution or ZSR represents solution of the differential or difference equation for the given input.
- To obtain the particular solution or ZSR, one would have to use the method of integrating factors.
- y_p is not unique.
- Usually it is obtained by assuming an output of the same general form as the input.
- If $x[n] = \alpha_n$ then assume $y_p[n] = c\alpha^n$ and find the constant c so that $y_p[n]$ is the solution of given equation

1.1.3 Examples

Example 1 (ZSR)

- Consider the first order recursive system described by the difference equation $y[n]-\rho y[n-1] = x[n]$, find the particular solution when $x[n] = (1/2)^n$.
- Assume a particular solution of the form $y_p[n] = c_p(1/2)^n$.
- Put the values of $y_p[n]$ and $x[n]$ in the equation then we get $c_p(\frac{1}{2})^n - \rho c_p(\frac{1}{2})^{n-1} = (\frac{1}{2})^n$

- Multiply both the sides of the equation by $(1/2)^n$ we get $c_p = 1/(1-2\rho)$.
- Then the particular solution is

$$y_p[n] = \frac{1}{1-2\rho}(\frac{1}{2})^n$$

- For $\rho = (1/2)$ particular solution has the same form as the homogeneous solution
- However no coefficient c_p satisfies this condition and we must assume a particular solution of the form $y_p[n] = c_p n(1/2)^n$.
- Substituting this in the difference equation gives $c_p n(1-2\rho) + 2\rho c_p = 1$
- Using $\rho = (1/2)$ we find that $c_p = 1$.

Example 2 (ZSR)

- Consider the RC circuit described by $y(t) + RC\frac{d}{dt}y(t) = x(t)$
- Assume a particular solution of the form $y_p(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$.
- Replacing $y(t)$ by $y_p(t)$ and $x(t)$ by $\cos(\omega_0 t)$ gives

$$c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) - RC\omega_0 c_1\sin(\omega_0 t) + RC\omega_0 c_2\cos(\omega_0 t) = \cos(\omega_0 t)$$

- The coefficients c_1 and c_2 are obtained by separately equating the coefficients of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, gives

$$c_1 = \frac{1}{1+(RC\omega_0)^2} \qquad and \qquad c_2 = \frac{RC\omega_0}{1+(RC\omega_0)^2}$$

- Then the particular solution is

$$y_p(t) = \frac{1}{1+(RC\omega_0)^2}\cos(\omega_0 t) + \frac{RC\omega_0}{1+(RC\omega_0)^2}\sin(\omega_0 t)$$

Complete solution

- Find the form of the homogeneous solution y_h from the roots of the characteristic equation
- Find a particular solution y_p by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution
- Determine the coefficients in the homogeneous solution so that the complete solution $y = y_h + y_p$ satisfies the initial conditions

1.1.4 Unsolved example from [2]

Unsolved ex. 2.53

Determine the homogeneous solution of the system described by the differential equation

- (a) $5\frac{d}{dt}y(t) + 10y(t) = 2x(t)$
  Solution is
  $5r + 10 = 0$
  $r = -2$
  $y_h(t) = c_1 e^{-2t}$
- (b) $\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = \frac{d}{dt}x(t)$
  Solution is

$r^2 + 6r + 8 = 0$

$r = -4, -2$ and $y_h(t) = c_1 e^{-4t} + c_2 e^{-2t}$

Unsolved ex. 2.53

Determine the homogeneous solution of the system described by the differential equation

- (c) $\frac{d^2}{dt^2}y(t) + 4y(t) = 3\frac{d}{dt}x(t)$

  Solution is

  $r^2 + 4 = 0$

  $r = \pm 2j$

  $y_h(t) = c_1 e^{-2jt} + c_2 e^{2jt}$

- (d) $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 2y(t) = x(t)$

  Solution is

  $r^2 + 2r + 2 = 0$

  $r = -1 \pm j$

  $y_h(t) = c_1 e^{(-1+j)t} + c_2 e^{(-1-J)t}$

- (e) $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t)$

  Solution is

  $r^2 + 2r + 1 = 0$

  $r = \pm 1$

  $y_h(t) = c_1 e^{-t} + c_2 t e^{t}$

Unsolved ex. 2.54

Determine the homogeneous solution of the system described by the difference equation

- (a) $y[n] - \alpha y[n-1] = 2x[n]$
  Solution is
  $r - \alpha = 0$
  $y_h[n] = c_1\alpha^n$
- (b) $y[n] - (1/4)y[n-1] - (1/8)y[n-2] = x[n] + x[n-1]$
  Solution is
  $r^2 - (1/4)r - (1/8) = 0$
  $r = (1/2), (1/4)$ and $y_h[n] = c_1(\frac{1}{2})^n + c_2(\frac{1}{4})^n$

Unsolved ex. 2.54

Determine the homogeneous solution of the system described by the difference equation

- (c) $y[n] - (9/16)y[n-2] = x[n-1]$
  Solution is
  $r^2 + 9/16 = 0$
  $r = \pm j\frac{3}{4}$
  $y_h[n] = c_1(j\frac{3}{4})^n + c_2(-j\frac{3}{4})^n$
- (d) $y[n] + y[n-1] - (1/4)y[n-2] = x[n] + 2x[n-1]$
  Solution is
  $r^2 + r + 1/4 = 0$
  $r = -\frac{1}{2}, -\frac{1}{2}$ and $y_h[n] = c_1(-\frac{1}{2})^n + c_2 n(\frac{1}{2})^n$

Unsolved ex. 2.55

Determine the particular solution of the system described by the differential equation

- (a) $5\frac{d}{dt}y(t)+10y(t)=2x(t)$ and (i) $x(t)=2$
  Solution is
  $y_p(t)=k$
  $10k=2*2$
  $k=2/5$
  $y_p(t)=2/5$

- (a) $5\frac{d}{dt}y(t)+10y(t)=2x(t)$ and (ii) $x(t)=e^{-t}$
  Solution is
  $y_p(t)=ke^{-t}$
  $-5ke^{-t}+10ke^{-t}=2e^{-t}$
  $k=2/5$
  $y_p(t)=2/5e^{-t}$

Unsolved ex. 2.55

Determine the particular solution of the system described by the differential equation

- (a) $5\frac{d}{dt}y(t)+10y(t)=2x(t)$ and (iii) $x(t)=\cos(3t)$
  Solution is
  $y_p(t)=A\cos(3t)+B\sin(3t)$
  $\frac{d}{dt}y_p(t)=-3A\sin(3t)+3B\cos(3t)$
  $5(-3A\sin(3t)+3B\cos(3t))+10A\cos(3t)+10B\sin(3t)=2\cos(3t)$
  $-15A+10B=0$

$10A + 15B = 2$

$A = 4/65$

$B = 6/65$

$y_p(t) = (4/65)\cos(3t) + (6/65)\sin(3t)$

- (b) $\frac{d^2}{dt^2}y(t) + 4y(t) = 3\frac{d}{dt}x(t)$ and (i) $x(t) = t$
  Solution is
  $y_p(t) = k_1 t + k_2$
  $4k_1 t + 4k_2 = 3$
  $k_1 = 0$ and $k_2 = 3/4$
  $y_p(t) = 3/4$

Unsolved ex. 2.56

Determine the particular solution of the system described by the difference equation

- (a) $y[n] - (2/5)y[n-1] = 2x[n]$ and (i) $x[n] = 2u[n]$
  Solution is
  $y_p[n] = ku[n]$
  $k = (2/5)k = 4$ and $k = 20/3$
  $y_p[n] = (20/3)u[n]$
- (a) $y[n] - (2/5)y[n-1] = 2x[n]$ and (ii) $x[n] = -(1/2)^n u[n]$
  Solution is
  $y_p[n] = k(1/2)^n u[n]$
  $k(1/2)^n - (2/5)(1/2)^n k = -2(1/2)^n$ and $k = -10$
  $y_p[n] = -10(1/2)^n u[n]$

Unsolved ex. 2.57

Determine the output of the systems described by the following differential equations with input and initial conditions as specified:

- (a) $\frac{d}{dt}y(t) + 10y(t) = 2x(t)$, $y(0^-) = 1$, $x(t) = u(t)$

  Solution is

  $t \geq 0$ natural: characteristic equation

  $r + 10 = 0$ and $r = -10$

  $y_h(t) = ce^{-10t}$

  Particular solution: $y_p(t) = ku(t) = (1/5)u(t)$

  $y(t) = (1/5) + ce^{-10t}$

  $y(0^-) = 1 = (1/5) + c$, $c = (4/5)$

  $k_1 = 0$ and $k_2 = 3/4$

  $y(t) = (1/5)(1 + 4e^{-10t})u(t)$

Unsolved ex. 2.57

Determine the output of the systems described by the following difference equations with input and initial conditions as specified:

- (a) $y[n] - (1/2)y[n-1] = 2x[n]$, $y[-1] = 3$, $x[n] = (-1/2)^n u[n]$

  Solution is

  Homogeneous solution: $n \geq 0$, $r - (1/2) = 0$, $y_p[n] = c(1/2)^n$

  Particular solution: $y_p[n] = k - 1/2)^n u[n]$

  $k(-1/2)^n - (1/2)k(-1/2)^{n-1} = 2(-1/2)^n$ and $k = 1$

  $y_p[n] = (-1/2)^n u[n]$

  Initial conditions: $y[n] = (1/2)y[n-1] + 2x[n]$

$y[0] = (1/2)(3) + 2 = 7/2$
Total solution: $y[n] = (-1/2)^n u[n] + c(1/2)^n u[n]$
$7/2 = 1 + c$ and $c = 5/2$
$y[n] = (-1/2)^n u[n] + (5/2)(1/2)^n u[n]$

Example 1

- Find the solution of the first order recursive system described by the difference equation $y[n] - (1/4)y[n-1] = x[n]$ if $x[n] = (1/2)^n u[n]$ and the initial condition is $y[-1] = 8$.
- $y[n] = y_h[n] + y_p[n]$ (same example with $\rho = 1/4$)
- $y[n] = 2(\frac{1}{2})^n + c_1(\frac{1}{4})^n, \quad for \quad n \geq 0)$
- Coefficient c_1 is obtained from the initial conditions. First translate the initial condition to time $n = 0$ by rewriting equation in recursive form and put $n = 0$, which gives
- $y[0] = x[0] + (1/4)y[-1]$, which implies that $y[0] = 1 + (1/4) * 8 = 3$, putting $y[0] = 3$ in the $y[n]$ equation gives $3 = 2(\frac{1}{2})^0 + c_1(\frac{1}{4})^0$, which gives $c_1 = 1$
- The complete solution is $y[n] == 2(\frac{1}{2})^n + c_1(\frac{1}{4})^n \quad for \quad n \geq 0,$

Example 2

- Find the response of the RC circuit described by $y(t) + RC\frac{d}{dt}y(t) = x(t)$, to an input $x(t) = \cos(t)u(t)$. Assume $R = 1\Omega$ and $C = 1F$ and initial voltage across the capacitor is $y(0^-) = 2V$

- The homogeneous solution is $y_h(t) = ce^{-\frac{t}{RC}}$
- The particular solution is (with $\omega_0 = 1$)

$$y_p(t) = \frac{1}{1+(RC)^2}\cos(t) + \frac{RC}{1+(RC)^2}\sin(t)$$

- The complete solution is (assume with $\omega_0 = 1$, $R = 1$ and $C = 1F$)

$$y(t) = ce^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) \qquad t > 0$$

Conclusions

- Difference equation and differential equation
- Solution to difference equation and differential equation
- Homogeneous solution (ZIR) is due to initial conditions of the system and does not depend on the input
- Particular solution (ZSR) is due to input when initial conditions of the system are set to zero
- Total solution is a combination of homogeneous solution (ZIR) and particular solution (ZSR)

1.2 Class 2: System representations

Outline of today's class

- Characteristics of a system described by difference and differential system
- Block diagram representations
- Block diagram implementation of given systems represented by difference and differential equations

1.2.1 Characteristics

- Differential/difference equation consists of two terms
  1. Associated with the initial conditions: natural response
  2. Associated with only the input signal: forced response
- The complete output is: $y = y_n + y_f$

Natural response

- This is the system response when the input is zero
- Deals with
  1. dissipation of any stored energy
  2. memory of the past represented by the past inputs
- Since input is zero, the response can be obtained by homogeneous solution by choosing the coefficients so that the initial conditions are satisfied

- The natural response is determined without translating initial conditions forward in time

Ex 1: Natural response

- Consider the RC circuit described by $y(t) + RC\frac{d}{dt}y(t) = x(t)$, find the natural response of this system assuming $y(0) = 2\ V, R = 1\Omega$ and $C = 1F$.
- The homogeneous solution is $y_h(t) = c_1 e^{\frac{-t}{RC}} = c_1 e^{-t}\ V$
- The natural response is obtained by choosing c_1 so that the initial condition $y(0) = 2$ is satisfied.
- The initial condition implies that $c_1 = 2$
- Hence, the natural response is $y_h(t) = 2e^{-t}\ V \quad for\ \ t \geq 0$

Ex 2: Natural response

- Consider the first order recursive system described by the difference equation $y[n] - \frac{1}{4}y[n-1] = x[n]$, find the natural response with IC $y[-1] = 8$.
- The homogeneous solution for $N = 1$ is $y_h[n] = c_1(\frac{1}{4})^n$.
- Satisfaction of the IC $y[-1] = 8$ gives $8 = c_1(\frac{1}{4})^{-1}$, which gives $c_1 = 2$
- Hence the natural response is $y_n[n] = 2(\frac{1}{4})^n,\ \ n \geq -1$

Unsolved ex. 2.58(a)

Identify the natural response for the system: $\frac{d}{dt}y(t) + 10y(t) = 2x(t), y(0^-) = 1, x(t) = u(t)$

- $r+10=0$ and $r=-10$
  $y_n(t)=c_1e^{-10t}$
  $y(0^-)=1=c_1$
  $y_n(t)=e^{-10t}$

Unsolved ex. 2.58(c)

Identify the natural response for the system: $\frac{d^2}{dt^2}y(t)+6\frac{d}{dt}y(t)+8y(t)=2x(t)$, $y(0^-)=-1$, $\frac{d}{dt}y(t)|_{t=0^-}=1, x(t)=e^{-t}u(t)$

- $y_n(t)=c_1e^{-4t}u(t)+c_2e^{-2t}u(t)$
  $y(0^-)=-1=c_1+c_2$ and $\frac{d}{dt}y(t)|_{t=0^-}=1=-4c_1-2c_2$
  and $c_1=\frac{1}{2}, c_2=-\frac{3}{2}$
  $y_n(t)=\frac{1}{2}e^{-4t}-\frac{3}{2}e^{-2t}$

Forced response

- System response due to input signal assuming zero ICs
- Forced response is similar to complete solution with zero ICs; (i) system is *at rest* and (ii) no stored energy or memory
- Since the ICs are zero, the response is forced by the input signal when the system is at rest
- The forced response depend on particular solution, which is valid only for times $t>0$ and $n\geq 0$
- This means, the *at rest* conditions for discrete time system $y[-N]=0,\ldots$, must be translated forward to times $n=0,1,2\ldots$ to solve the undetermined coefficients.

Ex 1: Forced response

- Consider the first order recursive system described by the difference equation $y[n] - \frac{1}{4}y[n-1] = x[n]$, find the forced response if $x[n] = (\frac{1}{2})^n u[n]$.
- The complete solution is of the form $y[n] = 2(\frac{1}{2})^n + c_1(\frac{1}{4})^n, \quad n \geq 0$.
- The forced response is obtained by choosing c_1 so that the initial condition $y(-1) = 0$ is satisfied.
- The initial condition implies that $y(0) = x[0] + \frac{1}{4}y[-n]$, $y[0] = 1$, and $c_1 = -1$
- Hence, the forced response is $y_f[n] = 2(\frac{1}{2})^n - (\frac{1}{4})^n, \quad n \geq 0$

Ex 2: Forced response

Identify the forced response for the system: $RC\frac{d}{dt}y(t) + y(t) = x(t)$, $R = 1$, $C = 1$, $x(t) = \cos t u(t)$

- $y_f(t) = ce^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t, \quad t > 0$ $y(0^-) = y(0^+) = 0$ and $c = -\frac{1}{2}$
  $y_f(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t$

1.2.2 Unsolved examples from [2]

Unsolved ex. 2.58(a)

Identify the forced response for the system: $\frac{d}{dt}y(t) + 10y(t) = 2x(t)$, $y(0^-) = 1$, $x(t) = u(t)$

- $y_f(t) = \frac{1}{5} + ke^{-10t}$
  $y(0) = 0 = \frac{1}{5} + k$

and $k = -\frac{1}{5}$
$y_f(t) = \frac{1}{5} - \frac{1}{5}e^{-10t}$

Unsolved ex. 2.58(c)

Identify the forced response for the system: $\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = 2x(t)$, $y(0^-) = -1$, $\frac{d}{dt}y(t)|_{t=0^-} = 1, x(t) = e^{-t}u(t)$

- $y_f(t) = \frac{2}{3}e^{-t}u(t) + c_1e^{-4t}u(t) + c_2e^{-2t}u(t)$ $y(0) = 0 = \frac{2}{3} + c_1 + c_2$ and $\frac{d}{dt}y(t)|_{t=0^-} = 1 = \frac{2}{3} - 4c_1 - 2c_2$ and $c_1 = \frac{1}{3}, c_2 = -1$ $y_f(t) = \frac{2}{3}e^{-t}u(t) + \frac{1}{3}e^{-4t}u(t) - e^{-2t}u(t)$

Unsolved ex. 2.60(a)

Identify the forced response for the system: $y[n] - \frac{1}{2}y[n-1] = 2x[n], y[-1] = 3, x[n] = (-\frac{1}{2})^n u[n]$

- $y_f[n] = k(\frac{1}{2})^n + (-\frac{1}{2})^n$, Translate ICs
  $y[n] = \frac{1}{2}y[n-1] + 2x[n]$ $y[0] = (\frac{1}{2})(0) + 2 = 2 = k + 1$ and $k = 1$ $y_f[n] = (\frac{1}{2})^n + (-\frac{1}{2})^n$

Impulse response

- Solution to differential/difference equation can be used to find the impulse response
- The response to a system *at rest* is equivalent to step response of the system with zero ICs
- The impulse and step response are related by $h(t) = \frac{d}{dt}s(t)$ and $h[n] = s[n] - s[n-1]$, where $h(.)$ is the impulse response and $s(.)$ is the step response
- Impulse response in obtained by differentiating/differencing the step response
- No initial conditions for impulse response description
- Differential/difference equation representation is more flexible

Linearity and time invariance (TI)

- The forced response of an LTI system described by differential/difference equation is linear with respect to inputs
- Linear: Homogeneity and super position
- If $x_1 \rightarrow y_1^f$ and $x_2 \rightarrow y_2^f$ then $\alpha_1 x_1 + \alpha_2 x_2 \rightarrow \alpha_1 y_1^f + \alpha_2 y_2^f$
- Forced response is also causal: since the system is initially at rest
- Similarly, the natural response is linear: If y_1^n and y_2^n are two natural responses associated with ICs I_1 and I_2 then $\alpha_1 I_1 + \alpha_2 I_2 \rightarrow \alpha_1 y_1^n + \alpha_2 y_2^n$

- In general the response of an LTI system described by difference/differential equation is not time invariant, since the ICs will result in an output that does not shift with a time shift of the input

Roots of the characteristic eqn

- The forced response depends on both the input and the roots of the characteristic equation. It involves both homogeneous and particular solution.
- Basic form of the natural response is dependent entirely on the roots of the characteristic equation.
- Impulse response of an LTI system also depends on the roots of characteristic equation.
- Characteristic equation has considerable information about system behavior
- Stability of an LTI system are directly related to the roots: Output bounded for any set of ICs with zero input
- BI BO stability: each term of the natural response of system must be bounded
- LTI system is stable
  - Discrete case: $|r_i^n| < 1, \forall i$
  - Continuous case: $|e^{r_i t}|$ bounded, ie. $Re\{r_i\} < 0$,
  - System is on the verge of instability: if $|r_i| = 1$ or if $Re\{r_i\} = 0$

- LTI system is unstable
  - If any root of the characteristic equation has magnitude more than unity in discrete case
  - If the real part of any root of the characteristic equation is positive.
- The roots of characteristic eqn acts as indicators of the system behavior
- Stability condition states that natural response of an LTI system goes to zero as time approaches to infinity, since each term is decaying
- When natural response tends to zero then the system response should be zero for zero input when all the stored energy is dissipated
- Reinforces the LTI system behavior for zero input
- Response is also determined by the roots of the characteristic equation
- The characteristic equation is very important in both natural and forced response
- Once the natural response has decayed to zero, the system behavior is governed by the particular solution (input)
- The natural response describes the transient behavior of the system, used to find the time taken by the system to respond to a transient, is time it takes for natural response to decay to zero
- The natural response contains r_i^n for discrete and $e^{r_i t}$ for continuous time systems

- Response time depends on: (i) root of the characteristic equation with the largest magnitude in discrete case and (ii) root with largest real component in continuous case

1.2.3 Block diagram representations

- A block diagram is an interconnection of elementary operations that act on the input signal
- This method is more detailed representation of the system than impulse response or differential/difference equation representations
- The impulse response and differential/difference equation descriptions represent only the input-output behavior of a system, block diagram representation describes how the operations are ordered
- Each block diagram representation describes a different set of internal computations used to determine the system output
- Block diagram consists of three elementary operations on the signals:
  - Scalar multiplication: $y(t) = cx(t)$ or $y[n] = x[n]$, where c is a scalar
  - Addition: $y(t) = x(t) + w(t)$ or $y[n] = x[n] + w[n]$.
- Block diagram consists of three elementary operations on the signals:
  - Integration for continuous time LTI system: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ Time shift for discrete time LTI system: $y[n] = x[n-1]$
- Scalar multiplication: $y(t) = cx(t)$ or $y[n] = x[n]$, where c is a scalar

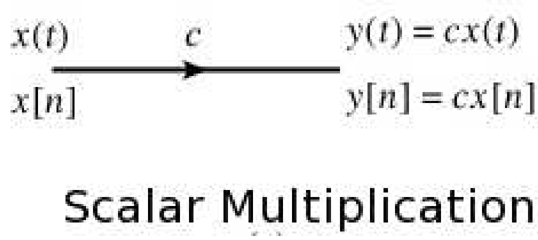


Figure 1.7: Scalar Multiplication

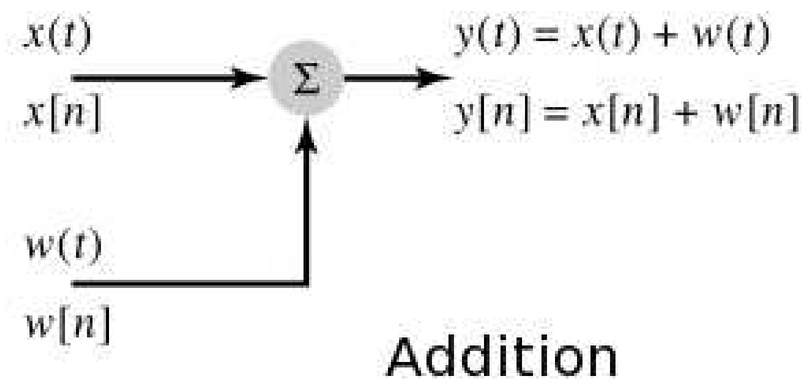


Figure 1.8: Addition

- Addition: $y(t) = x(t) + w(t)$ or $y[n] = x[n] + w[n]$
- Integration for continuous time LTI system: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
  Time shift for discrete time LTI system: $y[n] = x[n-1]$

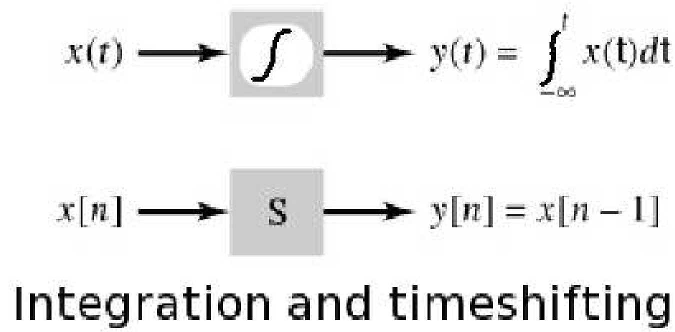


Figure 1.9: Integration and time shifting

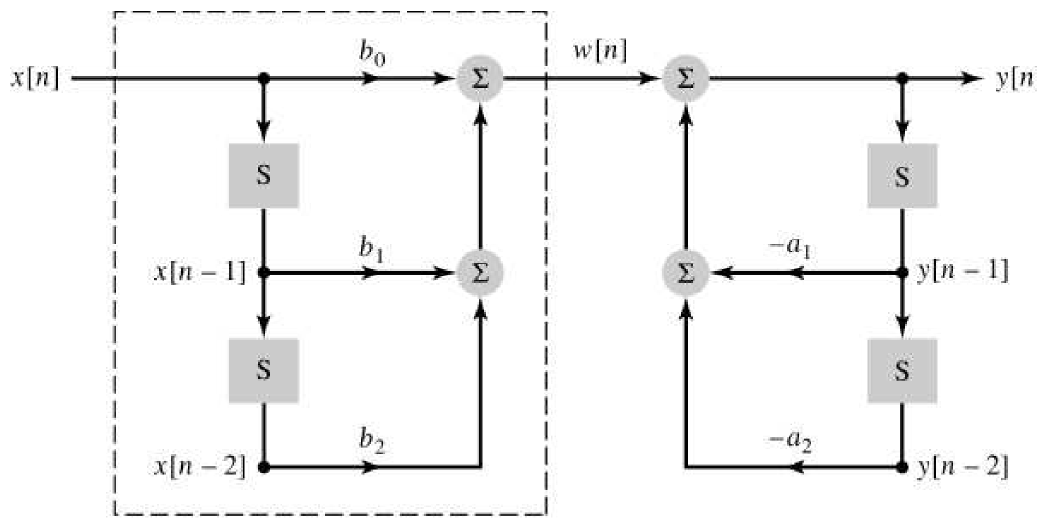


Figure 1.10: Example 1: Direct form I

1.2.4 Examples

Example 1

- Consider the system described by the block diagram as in Figure 1.10
- Consider the part within the dashed box
- The input $x[n]$ is time shifted by 1 to get $x[n-1]$ and again time shifted by one to get $x[n-2]$. The scalar multiplications are carried out and

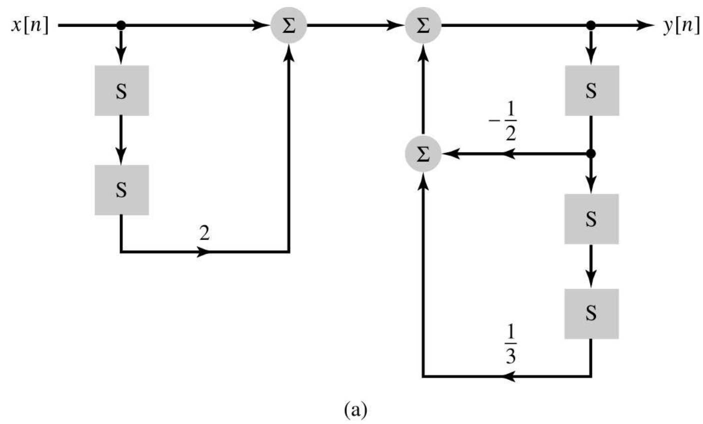


Figure 1.11: Example 2: Direct form I

they are added to get $w[n]$ and is given by

$$w[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2].$$

- Write $y[n]$ in terms of $w[n]$ as input $y[n] = w[n] - a_1 y[n-1] - a_2 y[n-2]$
- Put the value of $w[n]$ and we get $y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$
  and $y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$
- The block diagram represents an LTI system

Example 2

- Consider the system described by the block diagram and its difference equation is $y[n] + (1/2)y[n-1] - (1/3)y[n-3] = x[n] + 2x[n-2]$

Example 3

- Consider the system described by the block diagram and its difference equation is $y[n] + (1/2)y[n-1] + (1/4)y[n-2] = x[n-1]$

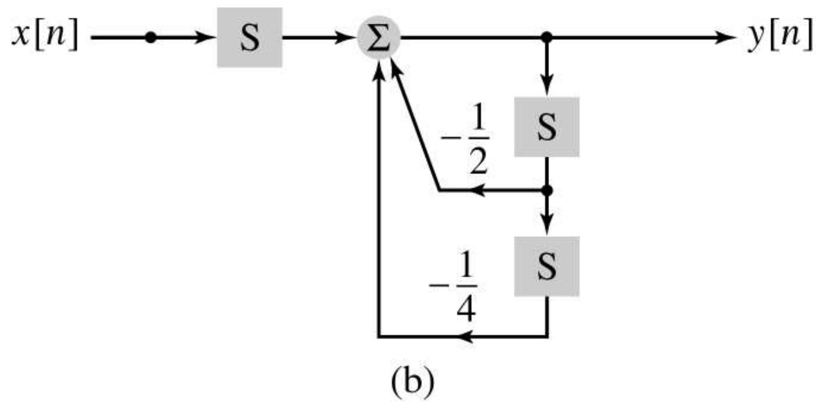


Figure 1.12: Example 3: Direct form I

Example 1 contd..

- Block diagram representation is not unique, direct form II structure of Example 1
- We can change the order without changing the input output behavior Let the output of a new system be $f[n]$ and given input $x[n]$ are related by

$$f[n] = -a_1 f[n-1] - a_2 f[n-2] + x[n]$$

- The signal $f[n]$ acts as an input to the second system and output of second system is

$$y[n] = b_0 f[n] + b_1 f[n-1] + b_2 f[n-2].$$

- The block diagram representation of an LTI system is not unique

1.2.5 Unsolved examples from [2]

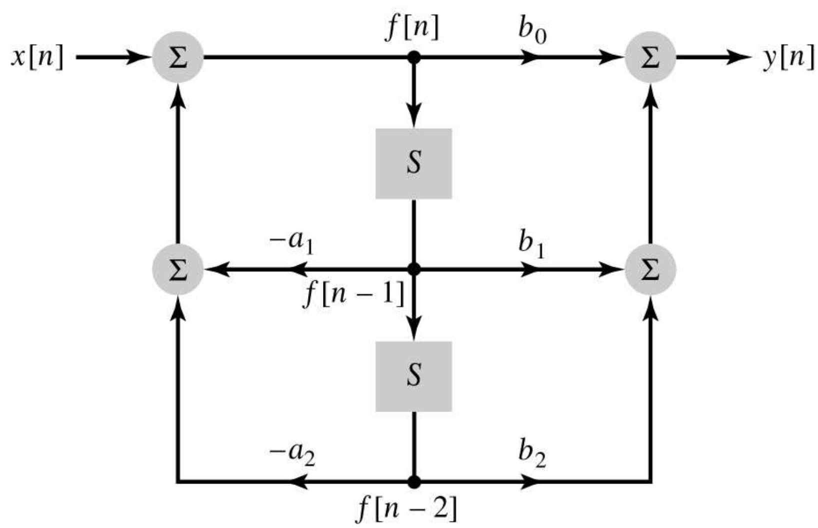


Figure 1.13: Example 1: Direct form II

Unsolved example 2.65(a)

- Find the difference equation for the system
- $f[n] = -2y[n] + x[n], y[n] = f[n-1] + 2f[n]$ $y[n] = -2y[n-1] + x[n-1] - 4y[n] + 2x[n]$ $5y[n] + 2y[n-1] = x[n-1] + 2x[n]$

Unsolved example 2.65(b)

- Find the difference equation for the system
- $f[n] = y[n] + x[n-1], \qquad y[n] = f[n-1] = y[n-1] + x[n-2]$

Unsolved example 2.65(c)

- Find the difference equation for the system
- $f[n] = x[n] - (1/8)y[n], y[n] = x[n-1] + f[n-2]$ $y[n] + (1/8)y[n-2] = x[n-1] + x[n-2]$

Example 1: Direct form I

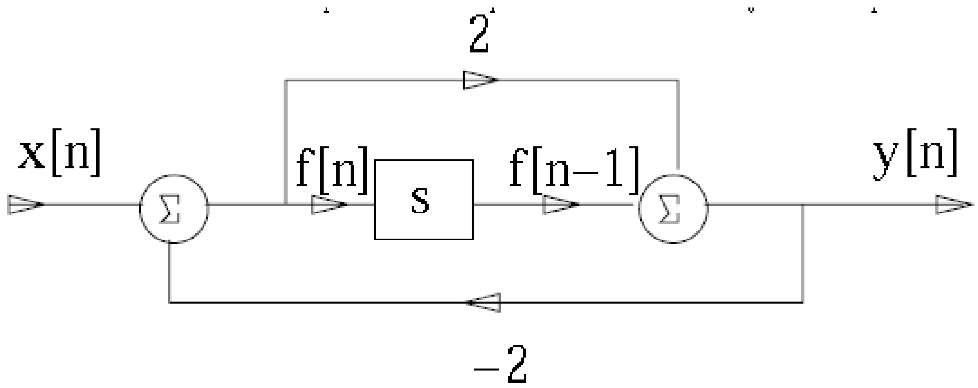


Figure 1.14: Unsolved example 2.65(a)

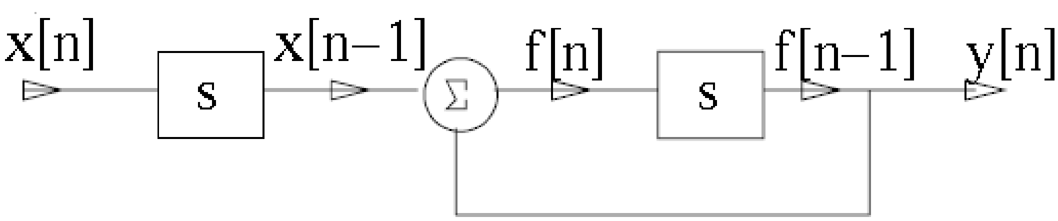


Figure 1.15: Unsolved example 2.65(b)

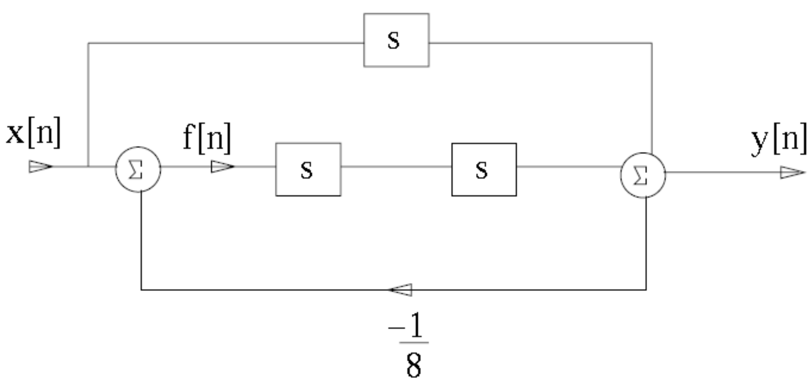


Figure 1.16: Unsolved example 2.65(c)

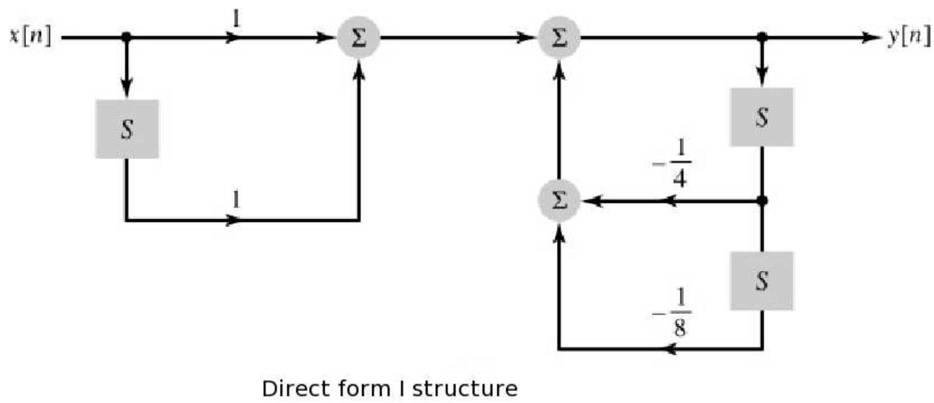


Figure 1.17: Example 1: Direct form I

- Draw the direct form I of the system $y[n]+(1/4)y[n-1]+(1/8)y[n-2]=x[n]+x[n-1]$

Example 1: Direct form II

- Draw the direct form II of the system $y[n]+(1/4)y[n-1]+(1/8)y[n-2]=x[n]+x[n-1]$

Unsolved ex.2.66(b): Direct form I

- Draw the direct form I if the system $y[n]+(1/2)y[n-1]-(1/8)y[n-2]=x[n]+2x[n-1]$

Unsolved ex.2.66(b): Direct form II

- Draw the direct form II of the system $y[n]+(1/2)y[n-1]-(1/8)y[n-2]=x[n]+2x[n-1]$

Unsolved ex.2.66(c): Direct form I

- Draw the direct form I of the system $y[n]+(1/2)y[n-1]-(1/8)y[n-2]=x[n]+2x[n-1]$

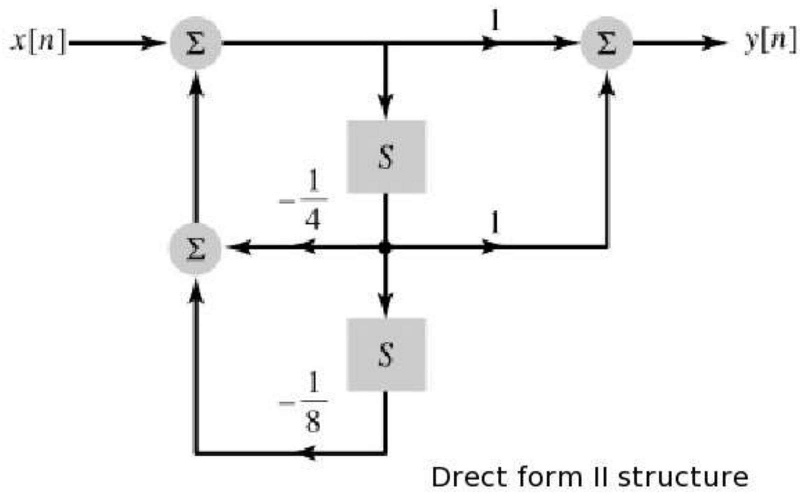


Figure 1.18: Example 1: Direct form II

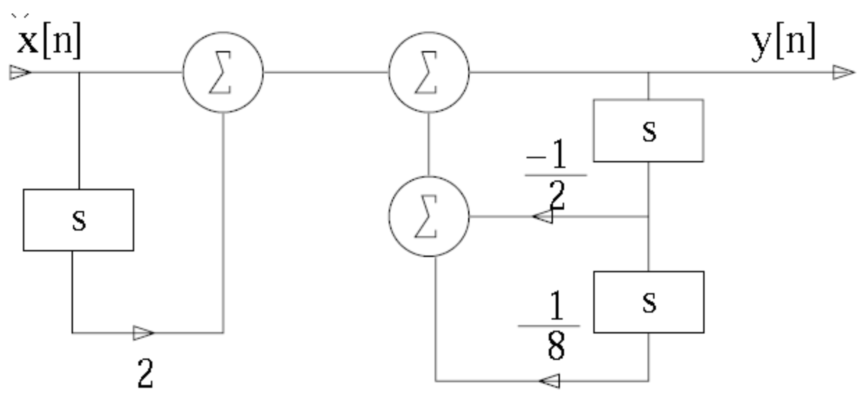


Figure 1.19: Unsolved example 2.66(b), direct form I

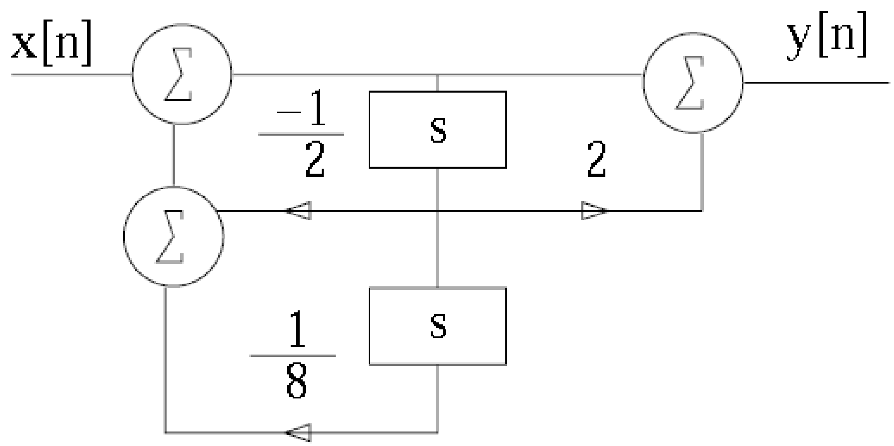


Figure 1.20: Unsolved example 2.66(b), direct form II

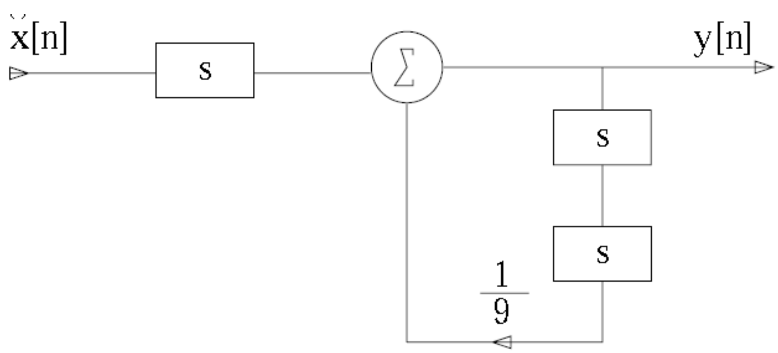


Figure 1.21: Unsolved example 2.66(c), direct form I

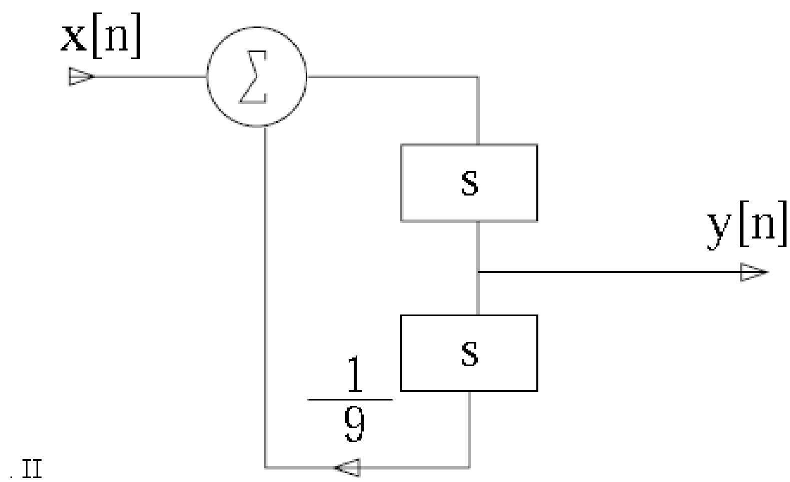


Figure 1.22: Unsolved example 2.66(c), direct form II

Unsolved ex.2.66(c): Direct form II

- Draw the direct form II of the system $y[n]+(1/2)y[n-1]-(1/8)y[n-2]=x[n]+2x[n-1]$

Unsolved ex.2.66(d): Direct form I

- Draw the direct form I of the system $y[n]+(1/2)y[n-1]-(1/8)y[n-2]=x[n]+2x[n-1]$

Unsolved ex.2.66(d): Direct form II

- Draw the direct form II of the system $y[n]+(1/2)y[n-1]-(1/8)y[n-2]=x[n]+2x[n-1]$

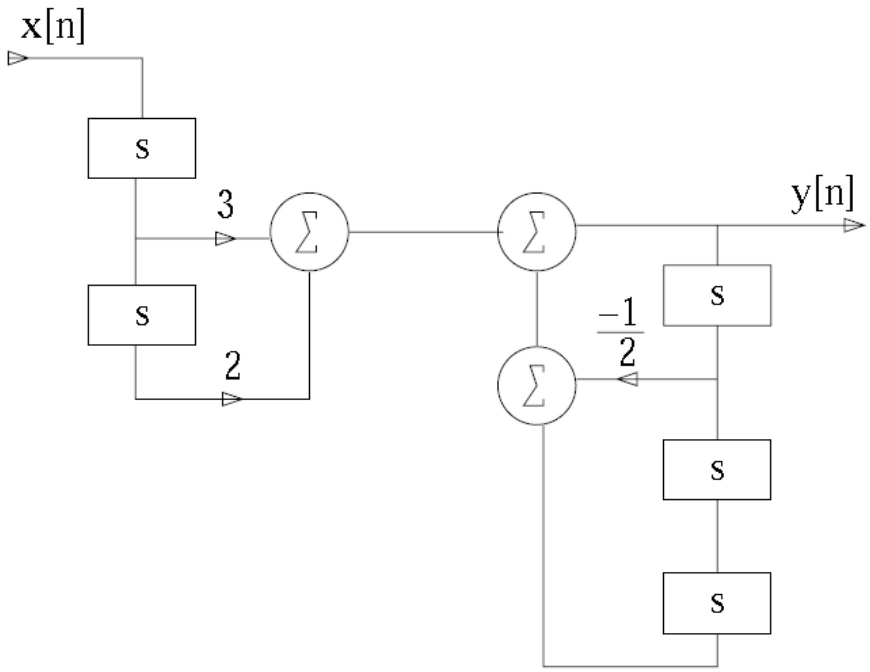


Figure 1.23: Unsolved example 2.66(d), direct form I

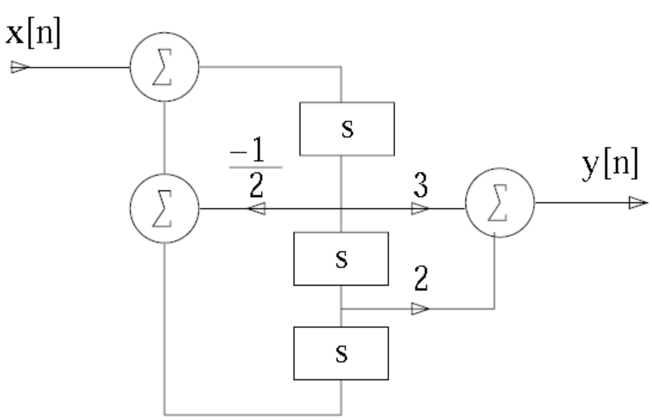


Figure 1.24: Unsolved example 2.66(d), direct form II

Continuous time

- Rewrite the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

  as an integral equation. Let $v^{(0)}(t) = v(t)$ be an arbitrary signal, and set

$$v^{(n)}(t) = \int_{-\infty}^{t} v^{(n-1)}(\tau) d\tau, \qquad n = 1, 2, 3, \ldots$$

  where $v^{(n)}(t)$ is the n-fold integral of $v(t)$ with respect to time

- Rewrite in terms of an initial condition on the integrator as

$$v^{(n)}(t) = \int_{0}^{t} v^{(n-1)}(\tau) d\tau + v^{(n)}(0), \;\; n = 1, 2, 3, \ldots$$

- If we assume zero ICs, then differentiation and integration are inverse operations, ie.

$$\frac{d}{dt} v^{(n)}(t) = v^{(n-1)}(t), \;\; t > 0 \;\text{ and }\; n = 1, 2, 3, \ldots$$

- Thus, if $N \geq M$ and integrate N times, we get the integral description of the system

$$\sum k = 0^N a_k y^{(N-k)}(t) = \sum k = 0^M b_k x^{(N-k)}(t)$$

- For second order system with $a_0 = 1$, the differential equation can be

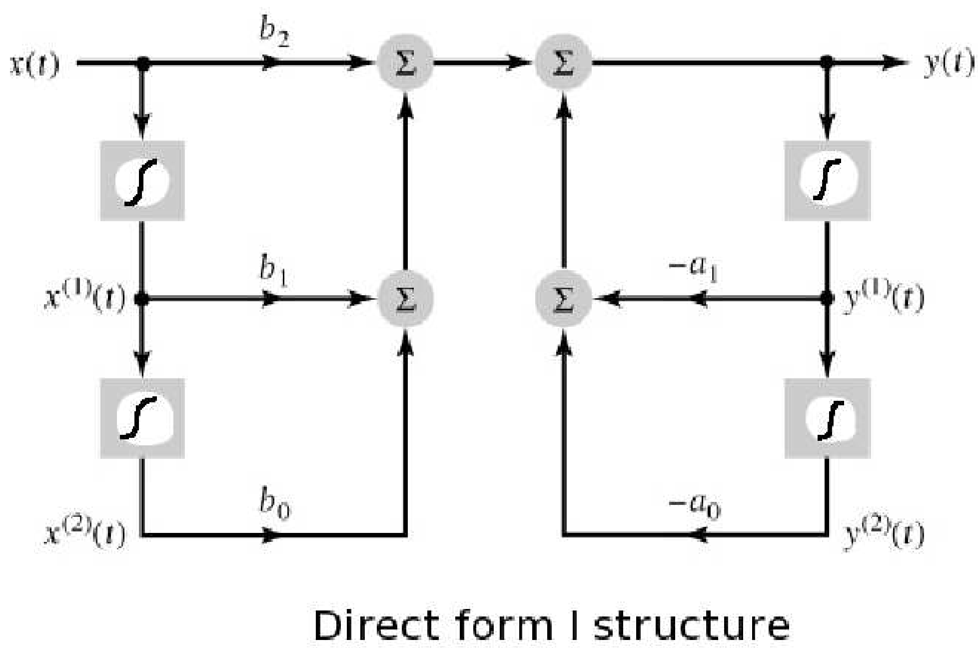


Figure 1.25: Direct form I

written as

$$y(t) = -a_1 y^{(1)}(t) - a_0 y^{(2)}(t) + b_2 x(t) + a_1 x^{(1)}(t) + b_0 x^{(2)}(t)$$

continuous time Direct form I **continuous time Direct form II**

Unsolved ex.2.67(b): Direct form I

- Draw the direct form I of the system $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 4y(t) = \frac{d}{dt}x(t)$

Unsolved ex.2.67(b): Direct form II

- Draw the direct form II of the system $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 4y(t) = \frac{d}{dt}x(t)$

Unsolved ex.2.67(c): Direct form I

- Draw the direct form I of the system $\frac{d^2}{dt^2}y(t) + y(t) = 3\frac{d}{dt}x(t)$

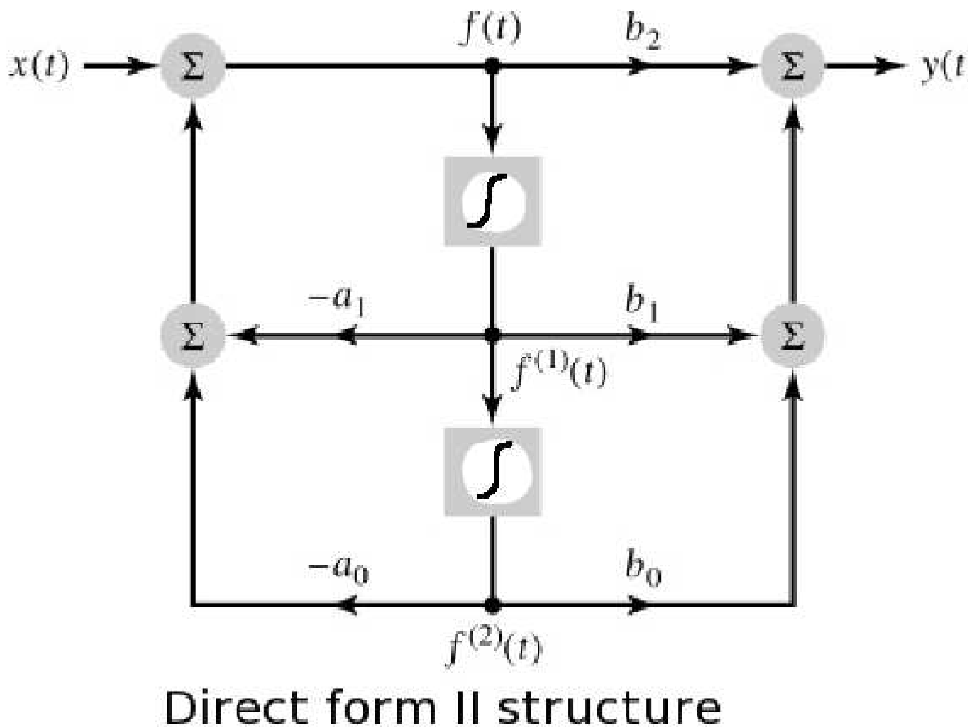


Figure 1.26: Direct form II

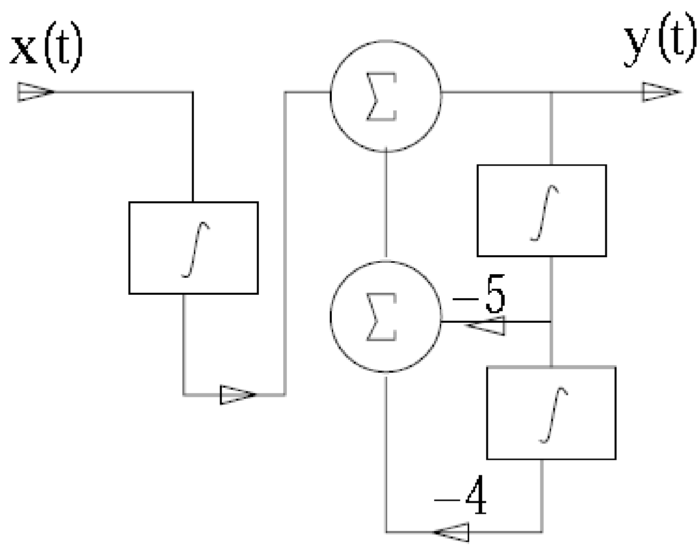


Figure 1.27: Unsolved example 2.67(b), direct form I

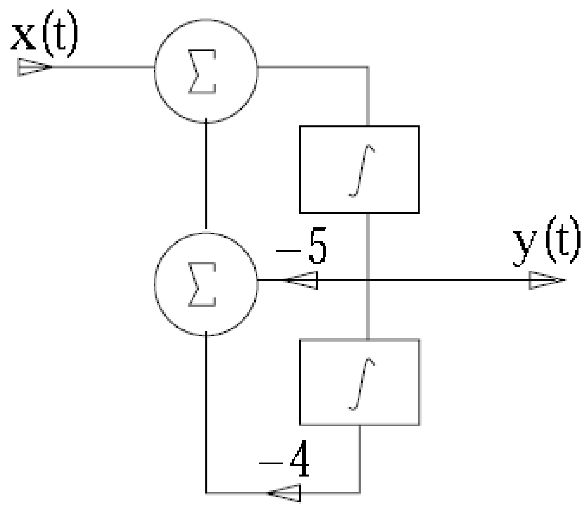


Figure 1.28: Unsolved example 2.67(b), direct form II

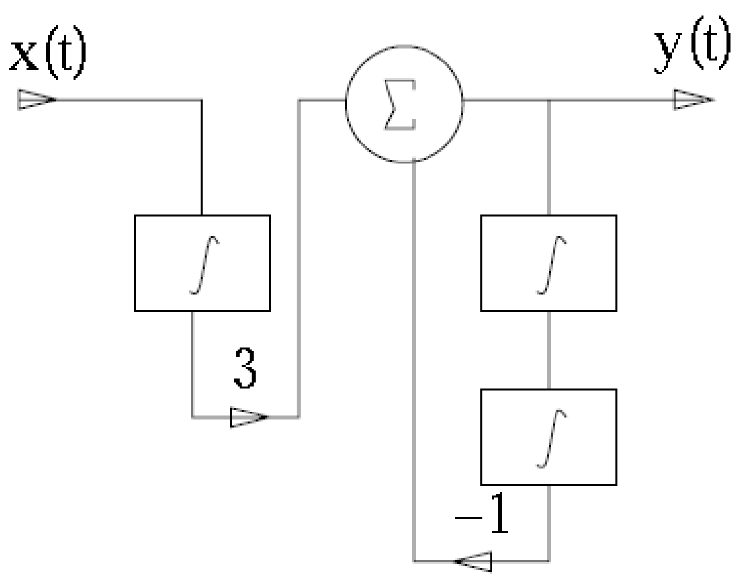


Figure 1.29: Unsolved example 2.67(c), direct form I

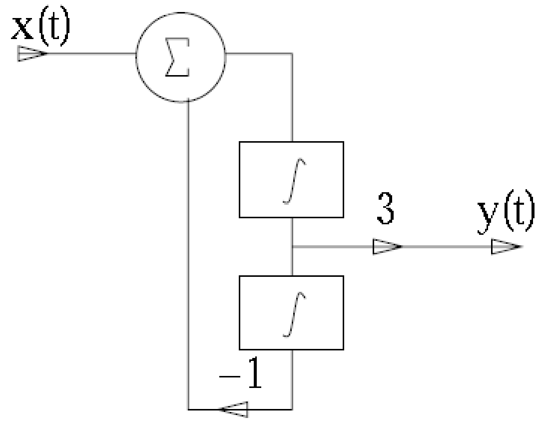


Figure 1.30: Unsolved example 2.67(c), direct form II

Unsolved ex.2.67(c): Direct form II

- Draw the direct form II of the system $\frac{d^2}{dt^2}y(t) + y(t) = 3\frac{d}{dt}x(t)$

Extra problems: Natural response

- $y(0) = 3, \frac{d}{dt}y(t)|_{t=0} = -7$: $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 2x(t) + \frac{d}{dt}x(t)$
- $y(0) = 0, \frac{d}{dt}y(t)|_{t=0} = -1$: $\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t) + \frac{d}{dt}x(t)$
- $y[-1] = \frac{4}{3}, y[-2] = \frac{16}{3}$: $y[n] - (\frac{9}{16}y[n-2] = x[n-1]$
- $y[0] = 2, y[1] = 0$: $y[n] + \frac{1}{4}y[n-2] = x[n] + 2x[n-2]$

Conclusions

- Characteristics of a system described by difference and differential system: responses, roots of characteristic equation and linearity and time invariance
- Block diagram representations of systems
- Block diagram implementation of given systems represented by difference and differential equations

1.3 Class 3: z-Transform

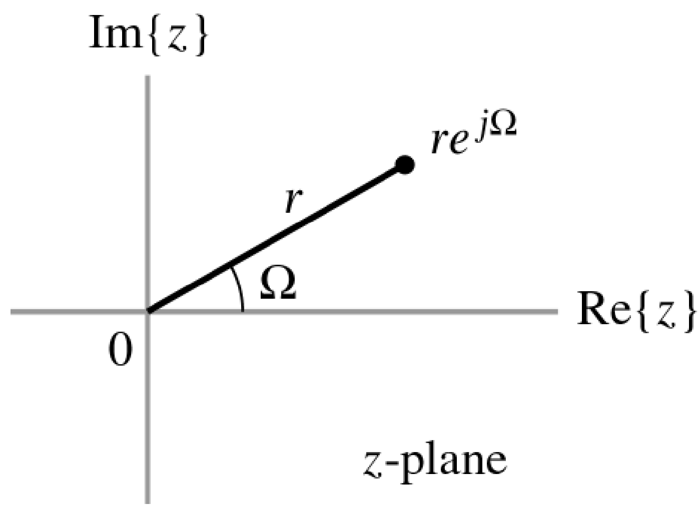
Outline of today's class

- Introduction to z-transform
- The z-plane
- The z-transform
- Convergence
- Poles and zeros

1.3.1 Introduction to z-transform

The z-transform is a transform for sequences. Just like the Laplace transform takes a function of t and replaces it with another function of an auxiliary variable s. The z-transform takes a sequence and replaces it with a function of an auxiliary variable, z. The reason for doing this is that it makes difference equations easier to solve, again, this is very like what happens with the Laplace transform, where taking the Laplace transform makes it easier to solve differential equations. A difference equation is an equation which tells you what the $k+2$th term in a sequence is in terms of the $k+1$th and kth terms, for example. Difference equations arise in numerical treatments of differential equations, in discrete time sampling and when studying systems that are intrinsically discrete, such as population models in ecology and epidemiology and mathematical modelling of mylinated nerves.

- Generalizes the complex sinusoidal representations of DTFT to more generalized representation using complex exponential signals



- It is the discrete time counterpart of Laplace transform

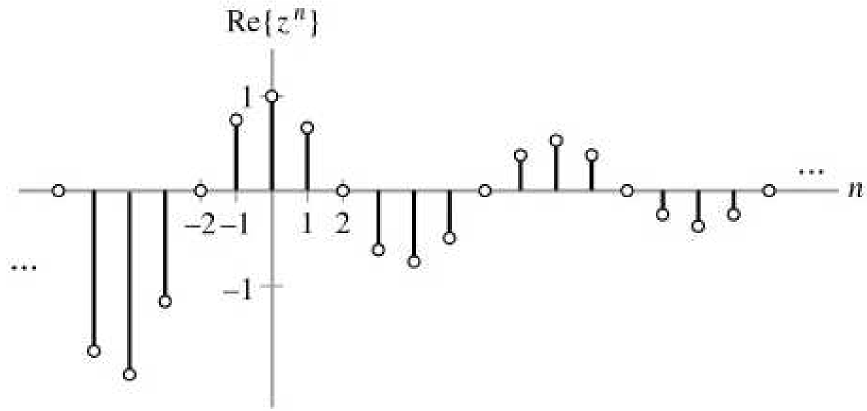
The *z*-Plane

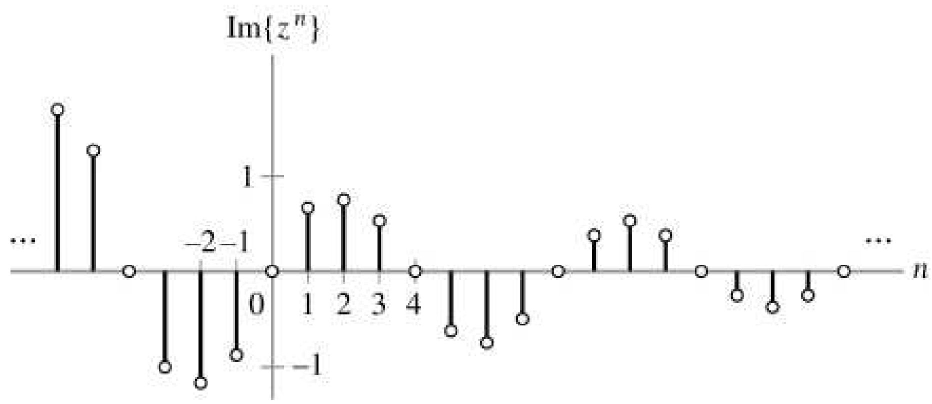
- Complex number $z = re^{j\Omega}$ is represented as a location in a complex plane (*z-plane*)

The *z*-transform

- Let $z = re^{j}\Omega$ be a complex number with magnitude r and angle Ω.
- The signal $x[n] = z^n$ is a complex exponential and $x[n] = r^n \cos(\Omega n) + jr^n \sin(\Omega n)$
- The real part of $x[n]$ is exponentially damped cosine
- The imaginary part of $x[n]$ is exponentially damped sine
- Apply $x[n]$ to an LTI system with impulse response $h[n]$, Then

$$y[n] = H\{x[n]\} = h[n] * x[n]$$


Re{z^n}
1
−2 −1
1 2
−1
n
...
...


Im{z^n}
1
−2 −1
0
1 2 3 4
−1
n
...
...

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- If

$$x[n] = z^n$$

  we get

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k}$$

$$y[n] = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- The z-transform is defined as

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

  we may write as

$$H(z^n) = H(z)z^n$$

You can see that when you do the z-transform it sums up all the sequence, and so the individual terms affect the dependence on z, but the resulting function is just a function of z, it has no k in it. It will become clearer later why we might do this.

- This has the form of an eigen relation, where z^n is the eigen function and $H(z)$ is the eigenvalue.

- The action of an LTI system is equivalent to multiplication of the input by the complex number $H(z)$.

- If $H(z) = |H(z)|e^{j\phi(z)}$ then the system output is

$$y[n] = |H(z)|e^{j\phi(z)}z^n$$

- Using $z = re^{j\Omega}$ we get

$$y[n] = |H(re^{j\Omega})|r^n \cos(\Omega n + \phi(re^{j\Omega})+$$

$$j|H(re^{j\Omega})|r^n \sin(\Omega n + \phi(re^{j\Omega})$$

- Rewriting $x[n]$

$$x[n] = z^n = r^n \cos(\Omega n) + jr^n \sin(\Omega n)$$

- If we compare $x[n]$ and $y[n]$, we see that the system modifies
  - the amplitude of the input by $|H(re^{j\Omega})|$ and
  - shifts the phase by $\phi(re^{j\Omega})$

DTFT and the z-transform

- Put the value of z in the transform then we get

$$H(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n](re^{j\Omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (h[n]r^{-n})e^{-j\Omega n}$$

- We see that $H(re^{j\Omega})$ corresponds to DTFT of $h[n]r^{-n}$.
- The inverse DTFT of $H(re^{j\Omega})$ must be $h[n]r^{-n}$.

- We can write

$$h[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})e^{j\Omega n}d\Omega$$

The z-transform contd..

- Multiplying $h[n]r^{-n}$ with r^n gives

$$h[n] = \frac{r^n}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})e^{j\Omega n}d\Omega$$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})(re^{j\Omega})^n d\Omega$$

- We can convert this equation into an integral over z by putting $re^{j\Omega} = z$
- Integration is over Ω, we may consider r as a constant
- We have

$$dz = jre^{j\Omega}d\Omega = jzd\Omega$$

$$d\Omega = \frac{1}{j}z^{-1}dz$$

- Consider limits on integral
  - Ω varies from $-\pi$ to π
  - z traverses a circle of radius r in a counterclockwise direction
- We can write $h[n]$ as $h[n] = \frac{1}{2\pi j}\oint H(z)z^{n-1}dz$
  where $\oint$ is integration around the circle of radius $|z| = r$ in a counter clockwise direction

- The *z-transform* of any signal $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- The *inverse z-transform* of is

$$x[n] = \frac{1}{2\pi j} \oint X(z)z^{n-1}dz$$

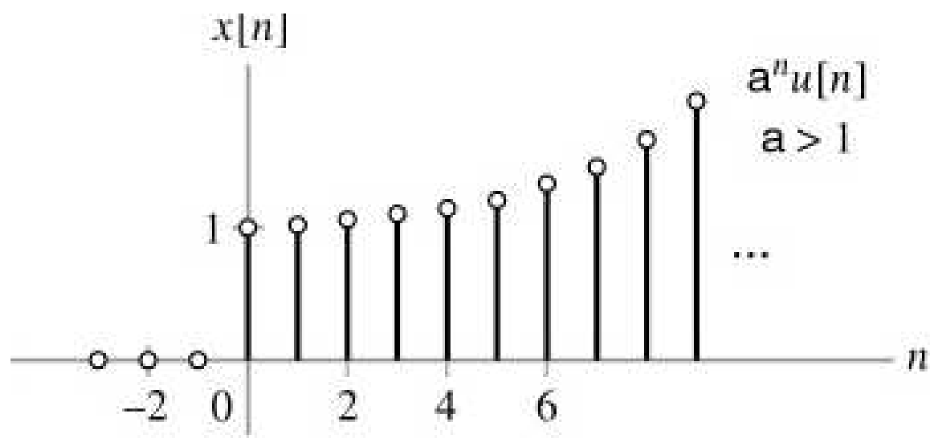
- *Inverse z-transform* expresses $x[n]$ as a weighted superposition of complex exponentials z^n
- The weights are $(\frac{1}{2\pi j})X(z)z^{-1}dz$
- This requires the knowledge of complex variable theory

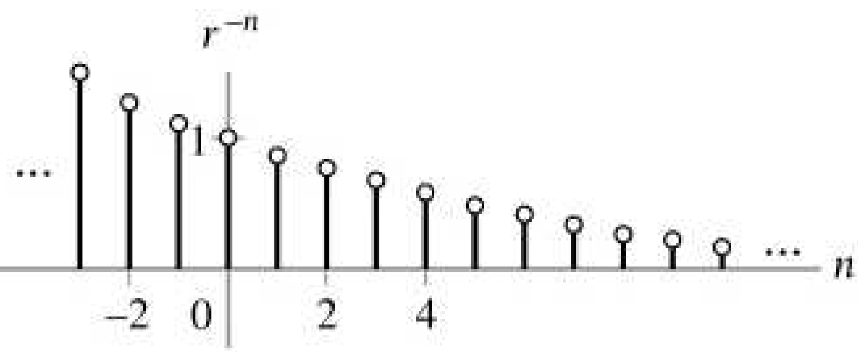
Convergence

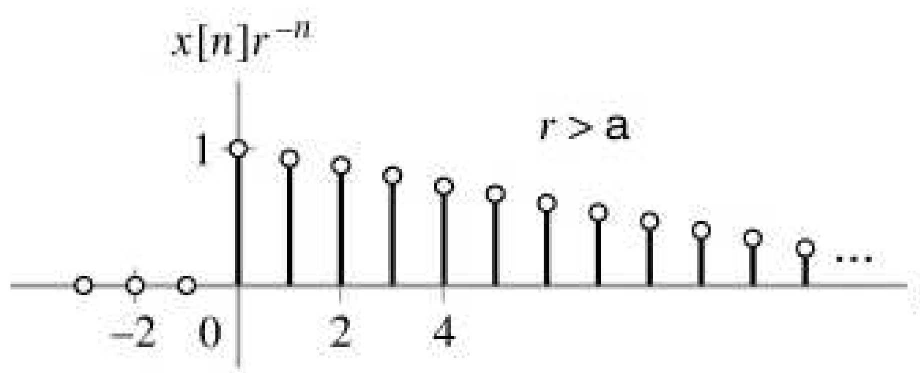
- Existence of *z-transform*: exists only if $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converges
- Necessary condition: absolute summability of $x[n]z^{-n}$, since $|x[n]z^{-n}| = |x[n]r^{-n}|$, the condition is

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

- The range r for which the condition is satisfied is called the *range of convergence* (ROC) of the *z-transform*
- ROC is very important in analyzing the system stability and behavior
- We may get identical *z-transform* for two different signals and only ROC differentiates the two signals
- The *z-transform* exists for signals that do not have DTFT.
- existence of DTFT: absolute summability of $x[n]$
- by limiting restricted values for r we can ensure that $x[n]r^{-n}$ is absolutely summable even though $x[n]$ is not
- Consider an example: the DTFT of $x[n] = \alpha^n u[n]$ does not exists for $|\alpha| > 1$
- If $r > \alpha$, then r^{-n} decays faster than $x[n]$ grows
- Signal $x[n]r^{-n}$ is absolutely summable and *z-transform* exists


x[n]
a^n u[n]
a > 1
1
...
n
–2
0
2
4
6


r^−n
...
1
...
n
–2
0
2
4


x[n]r^−n
1
r > a
...
–2
0
2
4

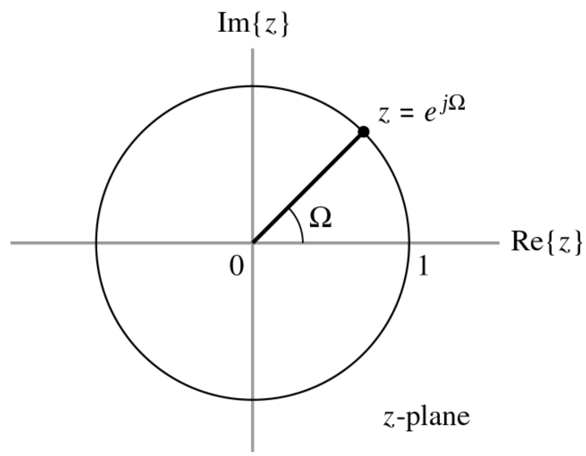


Figure 1.31: DTFT and z-transform

The z-Plane and DTFT

- If $x[n]$ is absolutely summable, then DTFT is obtained from the z-*transform* by setting $r = 1$ ($z = e^{j\Omega}$), ie. $X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}}$ as shown in Figure **??**

Poles and Zeros

- Commonly encountered form of the z-transform is the ratio of two polynomials in z^{-1}

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + b_N z^{-N}}$$

- It is useful to rewrite $X(z)$ as product of terms involving roots of the numerator and denominator polynomials

$$X(z) = \frac{\tilde{b} \prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

where $\tilde{b} = b_0/a_0$

Poles and Zeros contd..

- Zeros: The c_k are the roots of numerator polynomials
- Poles: The d_k are the roots of denominator polynomials
- Locations of zeros and poles are denoted by "$\bigcirc$" and "$\times$" respectively

Example 1

- The z-transform and DTFT of $x[n] = \{1, 2, -1, 1\}$ starting at $n = -1$
- $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-1}^{2} x[n]z^{-n} = z + 2 - z^{-1} + z^{-2}$
- $X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}} = e^{j\Omega} + 2 - e^{-j\Omega} + e^{-j2\Omega}$

- The z-transform and DTFT of $x[n] = \{1, 2, -1, 1\}$ starting at $n = -1$
- $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-1}^{2} x[n]z^{-n} = z + 2 - z^{-1} + z^{-2}$
- $X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}} = e^{j\Omega} + 2 - e^{-j\Omega} + e^{-j2\Omega}$

Example 2

- Find the z-transform of $x[n] = \alpha^n u[n]$, Depict the ROC and the poles and zeros
- Solution: $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} (\frac{\alpha}{z})^n$
  The series converges if $|z| > |\alpha|$
  $X(z) = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}, \qquad |z| > |\alpha|.$
  Hence pole at $z = \alpha$ and a zero at $z = 0$
- The ROC is

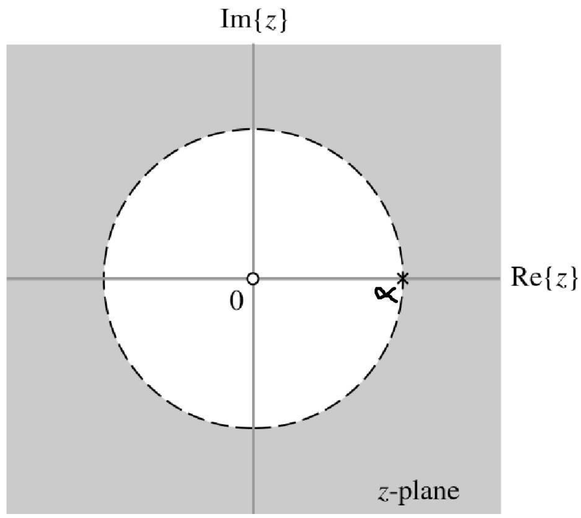


Figure 1.32: Example 2

Example 3

- Find the z-transform of $x[n] = -\alpha^n u[-n-1]$, Depict the ROC and the poles and zeros
- Solution: $X(z) = -\sum_{n=-\infty}^{-1}(\frac{\alpha}{z})^n = -\sum_{k=1}^{\infty}(\frac{z}{\alpha})^k = 1 - \sum_{k=0}^{\infty}(\frac{z}{\alpha})^k$
  The series converges if $|z| < |\alpha|$
  $X(z) = 1 - \frac{1}{1-z\alpha^{-1}} = \frac{z}{z-\alpha}, \qquad |z| < |\alpha|.$
  Hence pole at $z = \alpha$ and a zero at $z = 0$
- The ROC is

Example 4

- Find the z-transform of $x[n] = -u[-n-1] + (\frac{1}{2})^n u[n]$, Depict the ROC and the poles and zeros
- Solution: $X(z) = \sum_{n=-\infty}^{\infty}(\frac{1}{2})^n u[n] z^{-n} - u[-n-1]z^{-n}$
  $= \sum_{n=0}^{\infty}(\frac{1}{2z})^n - \sum_{n=-\infty}^{-1}(\frac{1}{z})^n \qquad = \sum_{n=0}^{\infty}(\frac{1}{2z})^n + 1 - \sum_{k=0}^{\infty} z^k$

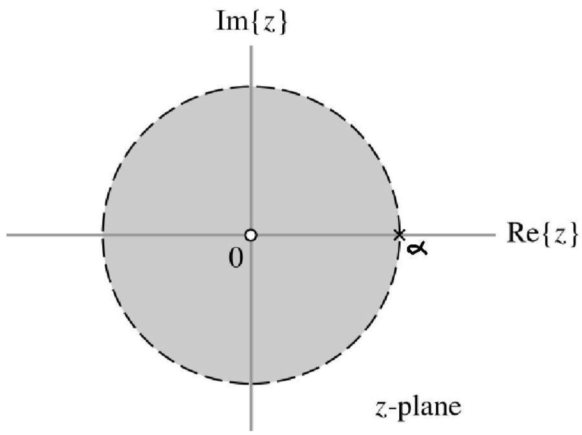


Figure 1.33: Example 3

- The series converges if $|z| > \frac{1}{2}$ and $|z| < 1$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + 1 - \frac{1}{1-z}, \quad 1/2 < |z| < 1.$$

$$X(z) = \frac{z(2z - \frac{3}{2})}{(z - \frac{1}{2})(z-1)}$$

Hence poles at $z = 1/2, \quad z = 1$ and zeros at $z = 0, \quad z = 3/4$

- The ROC is

1.3.2 Unsolved problems from [2]

Unsolved ex. 7.17a

- Find the z-transform of $x[n] = \delta[n-k]$, $k > 0$. Depict the ROC and the poles and zeros
- Solution: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = z^{-k}, \quad z \neq 0$
  Hence multiple poles at $z = 0$

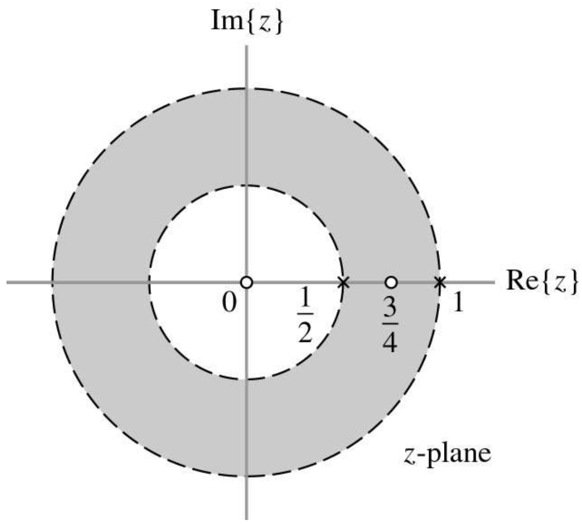


Figure 1.34: Example 4

Unsolved ex. 7.17b

- Find the z-transform of $x[n] = \delta[n+k]$, $k > 0$. Depict the ROC and the poles and zeros
- Solution: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = z^k, \quad all \;\; z$
  Hence multiple zeros at $z = 0$

Unsolved ex. 7.17d

- Find the z-transform of $x[n] = (\frac{1}{4})^n(u[n] - u[n-5])$.
- Solution: $X(z) = \sum_{n=0}^{4}(\frac{1}{4}z^{-1})^n = \frac{(z^5-(\frac{1}{4})^5)}{z^4(z-\frac{1}{4})} \quad all \;\; z$ Four poles at $z = 0$, and one pole at $z = 1/4$
  Five zeros at $z = \frac{1}{4}e^{jk\frac{2}{5}\pi}, \quad k = 0, 1, 2, 3, 4$
  Note that zero for $k = 0$ cancels the pole at $z = 1/4$

1.3.3 Exercises

Find the z-transforms of

$$\begin{array}{lll} (a)\ \left(\frac{1}{4^k}\right) & (b)\ (3^k) & (c)\ ((-2)^k) \\ (d)\ (4,16,64,256,\ldots) & (e)\ (1,-3,9,-27,\ldots) & (f)\ (0,1,4,12,64,160,\ldots) \end{array} \tag{1.4}$$

For (a) we have $r = 1/4$ so

$$\mathcal{Z}\left[\left(\frac{1}{4^k}\right)\right] = \frac{z}{z-1/4} = \frac{4z}{4z-1} \tag{1.5}$$

For (b) $r = 3$ giving For (a) we have $r = 1/4$ so

$$\mathcal{Z}[(3^k) = \frac{z}{z-3} \tag{1.6}$$

In (c) $r = -2$ but this makes no difference

$$\mathcal{Z}[((-2)^k)] = \frac{z}{z+2} \tag{1.7}$$

In (d) we see that $r = 4$ so

$$\mathcal{Z}[(4,16,64,256,\ldots)] = \frac{z}{z-4} \tag{1.8}$$

and in (e) $r = -3$ so

$$\mathcal{Z}[(1,-3,9,-27,\ldots)] = \frac{z}{z+3} \tag{1.9}$$

Finally, looking carefully at (f) you realize

$$(k2^{k-1}) = (0,1,4,12,64,160,\ldots) \tag{1.10}$$

and, hence,

$$\mathcal{Z}[(0,1,4,12,64,160,\ldots)] = \frac{z}{(z-2)^2} \tag{1.11}$$

Conclusions

- The z-transform and the z-plane
- Importance of ROC
- Relation between the DTFT and the z-transform
- Convergence of the z-transform
- Poles and zeros of $X(z)$

1.4 Class 4: Region of convergence (ROC)

Outline of today's class

- Region of convergence
- Properties ROC

The z-transform

- The *z-transform* of any signal $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- The *inverse z-transform* of is

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

Convergence

- Existence of *z-transform*: exists only if $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converges
- Necessary condition: absolute summability of $x[n]z^{-n}$, since $|x[n]z^{-n}| = |x[n]r^{-n}|$, the condition is

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

1.4.1 Properties of convergence

- ROC is related to characteristics of $x[n]$
- ROC can be identified from $X(z)$ and limited knowledge of $x[n]$
- The relationship between ROC and characteristics of the $x[n]$ is used to find inverse z-transform

Property 1

ROC can not contain any poles

- ROC is the set of all z for which z-transform converges
- $X(z)$ must be finite for all z
- If p is a pole, then $|H(p)| = \infty$ and z-transform does not converge at the pole
- Pole can not lie in the ROC

Property 2

The ROC for a finite duration signal includes entire z-plane except $z = 0$ or/and $z = \infty$

- Let $x[n]$ be nonzero on the interval $n_1 \leq n \leq n_2$. The z-transform is

$$X(z) = \sum_{n=n_1}^{n_2} x[n]z^{-n}$$

The ROC for a finite duration signal includes entire z-plane except $z = 0$ or/and $z = \infty$

- If a signal is causal ($n_2 > 0$) then $X(z)$ will have a term containing z^{-1}, hence ROC can not include $z = 0$
- If a signal is non-causal ($n_1 < 0$) then $X(z)$ will have a term containing powers of z, hence ROC can not include $z = \infty$

The ROC for a finite duration signal includes entire z-plane except $z = 0$ or/and $z = \infty$

- If $n_2 \leq 0$ then the ROC will include $z = 0$
- If $n_1 \geq 0$ then the ROC will include $z = \infty$
- This shows the only signal whose ROC is entire z-plane is $x[n] = c\delta[n]$, where c is a constant

Finite duration signals

- The condition for convergence is $|X(z)| < \infty$

$$|X(z)| = |\sum_{n=-\infty}^{\infty} x[n]z^{-n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]z^{-n}|$$

magnitude of sum of complex numbers $\leq$ sum of individual magnitudes

- Magnitude of the product is equal to product of the magnitudes

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}|$$

- split the sum into negative and positive time parts
- Let

$$I_-(z) = \sum_{n=-\infty}^{-1} |x[n]||z|^{-n}$$

$$I_+(z) = \sum_{n=0}^{\infty} |x[n]||z|^{-n}$$

- Note that $X(z) = I_-(z) + I_+(z)$. If both $I_-(z)$ and $I_+(z)$ are finite, then $X(z)$ if finite
- If $x[n]$ is bounded for smallest $+ve$ constants A_-, A_+, r_- and r_+ such that

$$|x[n]| \leq A_-(r_-)^n, \quad n < 0$$

$$|x[n]| \leq A_+(r_+)^n, \quad n \geq 0$$

- The signal that satisfies above two bounds grows no faster than $(r_+)^n$ for $+ve$ n and $(r_-)^n$ for $-ve$ n
- If the $n < 0$ bound is satisfied then

$$I_-(z) \leq A_- \sum_{n=-\infty}^{-1} (r_-)^n |z|^{-n}$$

$$= A_- \sum_{n=-\infty}^{-1} (\frac{r_-}{|z|})^n = A_- \sum_{k=1}^{\infty} (\frac{|z|}{r_-})^k$$

- Sum converges if $|z| \leq r_-$

- If the $n \geq 0$ bound is satisfied then

$$I_+(z) = A_+ \sum_{n=0}^{\infty} (r_+)^n |z|^{-n}$$

$$= A_+ \sum_{n=0}^{\infty} (\frac{r_+}{|z|})^n$$

- Sum converges if $|z| > r_+$
- If $r_+ < |z| < r_-$, then both $I_+(z)$ and $I_-(z)$ converge and $X(z)$ converges

To summarize:

- If $r_+ > r_-$ then no value of z for which convergence is guaranteed
- Left handed signal is one for which $x[n] = 0$ for $n \geq 0$
- Right handed signal is one for which $x[n] = 0$ for $n < 0$
- Two sided signal that has infinite duration in both +ve and -ve directions
- The ROC of a right-sided signal is of the form $|z| > r_+$

Finite duration signals contd..

- The ROC of a left-sided signal is of the form $|z| < r_-$
- The ROC of a two-sided signal is of the form $r_+ < |z| > r_-$

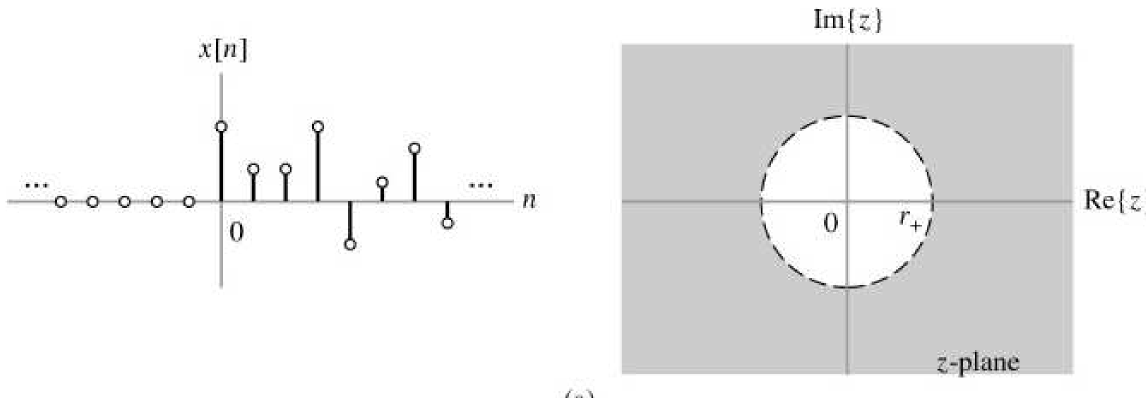


Figure 1.35: ROC of left sided sequence

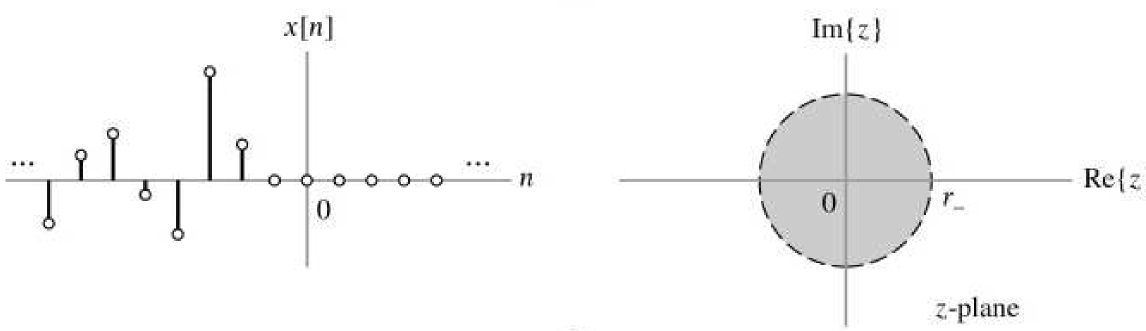


Figure 1.36: ROC of right sided sequence

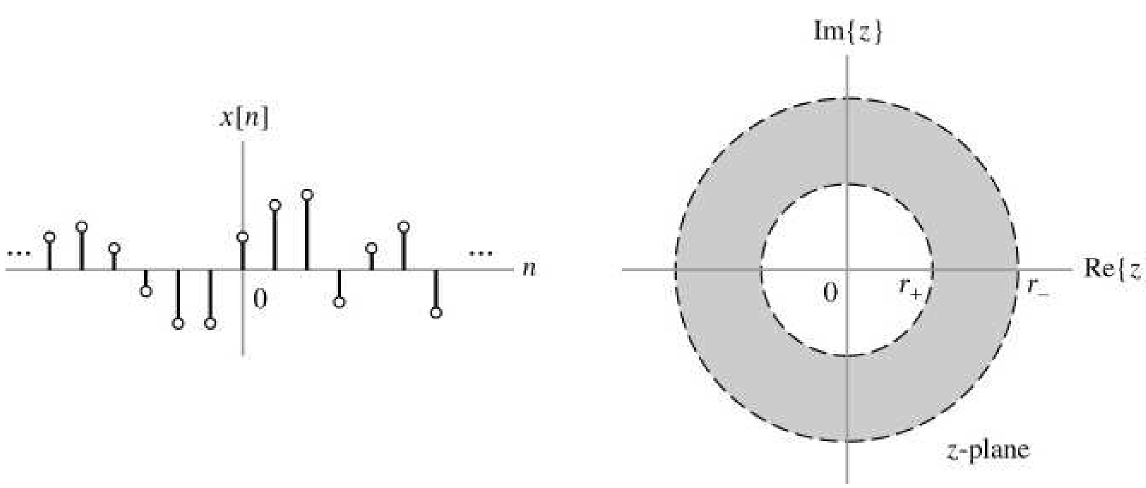


Figure 1.37: ROC of two sided sequence

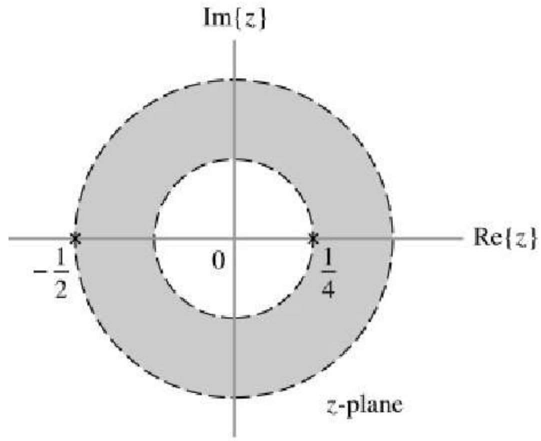


Figure 1.38: ROC of Example 1

1.4.2 Examples

Example 1

- Identify the ROC associated with the z-transform for $x[n] = (\frac{-1}{2})^n u[-n] + 2(\frac{1}{4})^n u[n]$
- The z-transform is $X(z) = \sum_{n=-\infty}^{0} (\frac{-1}{2z})^n + 2\sum_{n=0}^{\infty} (\frac{1}{4z})^n \quad = \sum_{k=0}^{\infty} (-2z)^k + 2\sum_{n=0}^{\infty} (\frac{1}{4z})^n$
- The first series converges for $|z| < \frac{1}{2}$, and second series converges for $|z| > \frac{1}{4}$.
- Both series must converge for $X(z)$ to converge, so the ROC is $\frac{1}{4} < |z| < \frac{1}{2}$, and $X(z)$ is

$$X(z) = \frac{1}{1+2z} + \frac{2z}{z-\frac{1}{4}}$$

- Poles are at $z = -(1/2)$ and $z = (1/4)$ and zero is at $z = 0$

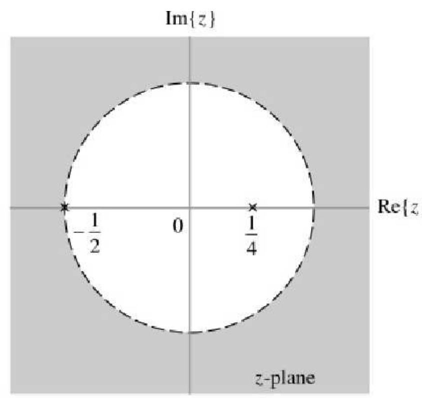


Figure 1.39: ROC of Example 2

Example 2

- Identify the ROC associated with the z-transform for $y[n] = (\frac{-1}{2})^n u[n] + 2(\frac{1}{4})^n u[n]$
- The z-transform is

$$Y(z) = \sum_{n=0}^{\infty} (\frac{-1}{2z})^n + 2\sum_{n=0}^{\infty} (\frac{1}{4z})^n$$

- The first series converges for $|z| > \frac{1}{2}$, and second series converges for $|z| > \frac{1}{4}$.
- Both series must converge for $Y(z)$ to converge, so the ROC is $|z| > \frac{1}{2}$, and $Y(z)$ is

$$Y(z) = \frac{z}{z + \frac{1}{2}} + \frac{2z}{z - \frac{1}{4}}$$

- Poles are at $z = -(1/2)$ and $z = (1/4)$ and zeros are at $z = 0$. The ROC is outside a circle containing the pole of largest radius $z = -1/2$

Example 3

- Identify the ROC associated with the z-transform for $w[n] = (\frac{-1}{2})^n u[-n] + 2(\frac{1}{4})^n u[-n]$
- The z-transform is

$$W(z) = \sum_{n=-\infty}^{0} (\frac{-1}{2z})^n + 2\sum_{n=-\infty}^{0} (\frac{1}{4z})^n$$

$$= \sum_{k=0}^{\infty} (-2z)^k + 2\sum_{k=0}^{\infty} (4z)^k$$

- The first series converges for $|z| < \frac{1}{2}$, and second series converges for $|z| < \frac{1}{4}$.
- Both series must converge for $W(z)$ to converge, so the ROC is $|z| < \frac{1}{4}$, and $W(z)$ is

$$W(z) = \frac{1}{1+2z} + \frac{2}{1-4z}$$

- Poles are at $z = -(1/2)$ and $z = (1/4)$ and zeros are at $z = 0$. The ROC is inside the a circle containing the pole of smallest radius $z = 1/4$

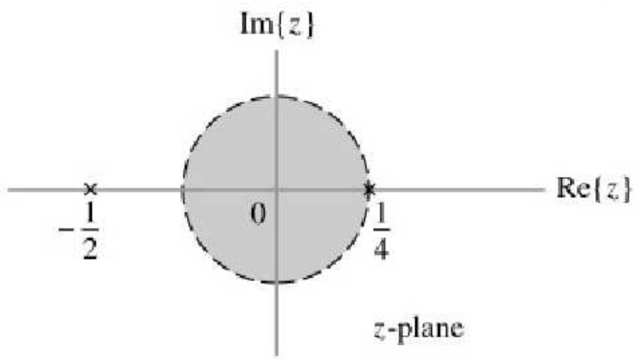


Figure 1.40: ROC of Example 3

Example 4

- Find z-transform and ROC for two sided signal $x[n] = \alpha^{|n|}$, for both $|\alpha| < 1$ and $|\alpha| > 1$
- The z-transform is

$$X(z) = \sum_{n=-\infty}^{-1} (\alpha^{-1}z^{-1})^n + \sum_{n=0}^{\infty} (\alpha/z)^n$$

$$= \sum_{k=0}^{\infty} (\alpha z)^k - 1 + \sum_{n=0}^{\infty} (\alpha/z)^n$$

- The first series converges for $|\alpha z| < 1$, ie. $|z| < \frac{1}{\alpha}$ and second series converges for $\frac{\alpha}{|z|} < 1$, ie. $|z| > \alpha$.
- Both series must converge for $X(z)$ to converge, so the ROC is $\alpha < |z| < \frac{1}{\alpha}$,

$$X(z) = \frac{1}{1-\alpha z} - 1 + \frac{1}{1-\frac{\alpha}{z}} = \frac{-z}{z-\frac{1}{\alpha}} + \frac{z}{z-\alpha}$$

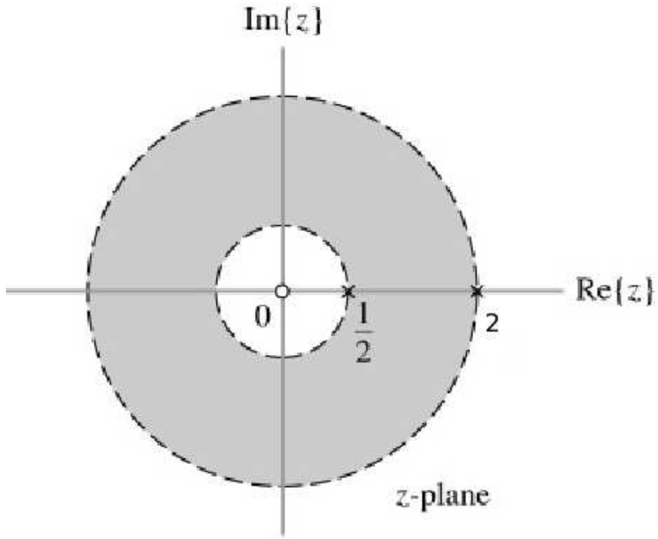


Figure 1.41: ROC of Example 4

- Poles are at $z = -(1/\alpha)$ and $z = \alpha$ and zeros are at $z = 0$. Let $\alpha = 1/2$
- What happens when $|\alpha| > 1$? The first series converges for $|\alpha z| < 1$, ie. $|z| < \frac{1}{\alpha}$ and second series converges for $\frac{\alpha}{|z|} < 1$, ie. $|z| > \alpha$.
- Both series must converge for $X(z)$ to converge, so the ROC is $\alpha < |z| < \frac{1}{\alpha}$, which is empty set

1.4.3 Unsolved examples from [2]

Unsolved ex. 7.18

- Given the z-transforms, determine whether the DTFT of the corresponding time signals exists without determining the time signal, and identify the DTFT in those cases where it exists:

Unsolved ex. 7.18(a)

- $X(z) = \frac{5}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$

- ROC includes $|z| = 1$, hence DTFT exists
- DTFT is

$$X(e^{j\Omega}) = \frac{5}{1 + \frac{1}{3}e^{-j\Omega}}$$

Unsolved ex. 7.18(b)

- $X(z) = \frac{5}{1+\frac{1}{3}z^{-1}}, \quad |z| < \frac{1}{3}$
- ROC does not include $|z| = 1$, hence DTFT does not exists

Unsolved ex. 7.18(c)

- $X(z) = \frac{z^{-1}}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}, \quad |z| < \frac{1}{2}$
- ROC does not include $|z| = 1$, hence DTFT does not exists

Unsolved ex. 7.18(d)

- $X(z) = \frac{z^{-1}}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}, \quad \frac{1}{2} < |z| < 3$
- ROC includes $|z| = 1$, hence DTFT exists
- DTFT is

$$X(z) = \frac{e^{-j\Omega}}{(1 - \frac{1}{2}e^{-j\Omega})(1 + 3e^{-j\Omega})}$$

Unsolved ex. 7.19

- The pole and zero locations of $X(z)$ are depicted in the z-plane on the following figures. In each case, identify all valid ROCs for $X(z)$ and specify the characteristics of the time signal corresponding to each ROC.

Unsolved ex. 7.19(a)

- Two poles at $z = -3/4$ and $z = 1/3$. Two zeros at $z = 0$ and $z = 3/2$ (Fig. P7.19(a))
- The z-transform is
$$X(z) = \frac{Cz(z - \frac{3}{2})}{(z + \frac{3}{4})(z - \frac{1}{3})}$$
- There are three possible ROCs
- There are three possible ROCs
- $|z| > \frac{3}{4}$, $x[n]$ is right sided sequence
- $\frac{1}{3} < |z| < \frac{3}{4}$, $x[n]$ is two sided sequence
- $|z| < \frac{1}{3}$, $x[n]$ is left sided sequence

Unsolved ex. 7.19(b)

- Three poles at $z = 0$, $z = 1 + j$ and $z = 1 - j$. Four zeros at $z = \pm 1$ and $z = \pm j$ (Fig. P7.19(b))
- The z-transform is
$$X(z) = \frac{C(z^4 - 1)}{z(z - \sqrt{2}e^{\frac{j\pi}{4}})(z - \sqrt{2}e^{\frac{-j\pi}{4}})}$$
- There are two possible ROCs
- $|z| > \sqrt{2}$, $x[n]$ is right sided sequence
- $|z| < \sqrt{2}$, $x[n]$ is two sided sequence

1.4.4 Exercises

Work out the z-transform of the following sequences

1. $(2, 4, 10, 28, \ldots)$
2. $(2, -2, 10, -26, \ldots)$
3. $(3, 0, 0, 0, \ldots)$
4. $(0, 0, 1, 1, 1, \ldots)$
5. $(0, 2, 4, 10, 28, \ldots)$
6. $(0, 0, 1, 2, 4, 8, \ldots)$
7. $(1, 1, 0, 1, 1, 1, \ldots)$

Answers

1. Consider the sequence

$$(2,4,10,28,\ldots) = (1,1,1,1,\ldots) + (1,3,9,27,\ldots) \tag{1.12}$$

use linearity

$$\begin{aligned} \mathcal{Z}[(2,4,10,28,\ldots)] &= \mathcal{Z}[(1,1,1,1,\ldots)] + \mathcal{Z}[(1,3,9,27,\ldots)] \\ &= \frac{z}{z-1} + \frac{z}{z-3} = \frac{2z^2-4z}{z^2-4z+3} \end{aligned} \tag{1.13}$$

2. Consider the sequence

$$(2,-2,10,-26,\ldots) = (1,1,1,1,\ldots) + (1,-3,9,-27,\ldots) \tag{1.14}$$

use linearity

$$\begin{aligned} \mathcal{Z}[(2,-2,10,-27,\ldots)] &= \mathcal{Z}[(1,1,1,1,\ldots)] + \mathcal{Z}[(1,-3,9,-27,\ldots)] \\ &= \frac{z}{z-1} + \frac{z}{z+3} = \frac{2z^2+2z}{z^2+2z-3} \end{aligned} \tag{1.15}$$

3. $(3,0,0,0,\ldots) = 3(\delta_k)$ so $\mathcal{Z}[(3,0,0,0,\ldots)] = 3$.

4. $(0,0,1,1,1,\ldots)$ is the $k_0 = 2$ delay of $(1,1,1,1,\ldots)$ which means that

$$\mathcal{Z}[(0,0,1,1,1,\ldots)] = \frac{1}{z^2}\frac{z}{z-1} \tag{1.16}$$

5. $(0,2,4,10,28,\ldots)$ is (x_{k-1}) where $(x_k) = (2,4,10,28,\ldots)$ as in first exercise, hence,

$$\mathcal{Z}[(0,2,4,10,28,\ldots)] = \frac{1}{z}\frac{2z^2-4z}{z^2-4z+3} = \frac{2z-4}{z^2-4z+3} \tag{1.17}$$

6. $(0,0,1,2,4,8,\ldots) = (2^{k-2})$ and $\mathcal{Z}[(2^k)] = z/(z-2)$, so,

$$\mathcal{Z}[(0,0,1,2,4,8,\ldots)] = \frac{1}{z^2}\frac{z}{z-2} = \frac{1}{z(z-2)} \tag{1.18}$$

7. This one is a bit trickier. Notice that

$$(1,1,0,1,1,1,\ldots) = (1,1,1,1,1,\ldots) - (0,0,1,0,0,\ldots) \tag{1.19}$$

and $(0,0,1,0,0,\ldots) = (\delta_{k-2})$. Hence, using linearity and the delay theorem we get

$$\begin{aligned} \mathcal{Z}[(1,1,0,1,1,1,\ldots)] &= \mathcal{Z}[(1,1,1,1,1,\ldots)] - \mathcal{Z}[(0,0,1,0,0,\ldots)] \\ &= \frac{z}{z-1} - \frac{1}{z^2} = \frac{z^3 - z + 1}{z^2(z-1)} \end{aligned} \tag{1.20}$$

Conclusions

- Region of convergence
- Properties ROC

1.5 Class 5: Properties of z-transform

Outline of today's class

- Properties of z-transform
  - Linearity
  - Time reversal
  - Time shift
  - Multiplication by α^n
  - Convolution
  - Differentiation in the z-domain

The z-transform

- The *z-transform* of any signal $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- The *inverse z-transform* of $X(z)$ is

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

Convergence

- Existence of *z-transform*: exists only if $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converges

- Necessary condition: absolute summability of $x[n]z^{-n}$, since $|x[n]z^{-n}| = |x[n]r^{-n}|$, the condition is

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

Properties of convergence

- ROC can not contain any poles
- The ROC for a finite duration signal includes entire z-plane except $z = 0$ or/and $z = \infty$

1.5.1 Properties of z-transform

- We assume that

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \qquad \text{with ROC} \quad R_x$$

$$y[n] \stackrel{z}{\longleftrightarrow} Y(z), \qquad \text{with ROC} \quad R_y$$

- General form of the ROC is a ring in the z-plane, so the effect of an operation on the ROC is described by the a change in the radii of ROC

P1: Linearity

- The z-transform of a sum of signals is the sum of individual z-transforms

$$ax[n] + by[n] \stackrel{z}{\longleftrightarrow} aX(z) + bY(z),$$

$$\text{with ROC at least} \quad R_x \cap R_y$$

- The ROC is the intersection of the individual ROCs, since the z-transform of the sum is valid only when both converge

P1: Linearity

- The ROC can be larger than the intersection if one or more terms in $x[n]$ or $y[n]$ cancel each other in the sum.
- Consider an example: $x[n] = (\frac{1}{2})^n u[n] - (\frac{3}{2})^n u[-n-1]$
- We have $x[n] \stackrel{z}{\longleftrightarrow} X(z)$

Pole-zero cancellation

- $X(z) = \frac{-z}{(z-\frac{1}{2})(z-\frac{3}{2})}$, with ROC $\frac{1}{2} < |z| < \frac{3}{2}$
- $y[n] = (\frac{1}{4})^n u[n] - (\frac{1}{2})^n u[n]$
- We have $y[n] \stackrel{z}{\longleftrightarrow} Y(z)$
- $Y(z) = \frac{-(\frac{1}{4})z}{(z-\frac{1}{4})(z-\frac{1}{2})}$, with ROC $|z| > \frac{1}{2}$
- Find z-transform of $ax[n] + by[n]$
- The pole-zero plot and ROC of $x[n]$
- The pole-zero plot and ROC of $y[n]$
- Linearity property indicates that $ax[n] + by[n] \stackrel{z}{\longleftrightarrow} a\frac{-z}{(z-\frac{1}{2})(z-\frac{3}{2})} + b\frac{-(\frac{1}{4})z}{(z-\frac{1}{4})(z-\frac{1}{2})}$
- In general ROC is the intersection of ROCs, ie. $\frac{1}{2} < |z| < \frac{3}{2}$
- However when $a = b$, the term $(\frac{1}{2})^n u[n]$ cancels in the time domain signal $ax[n] + by[n] =$

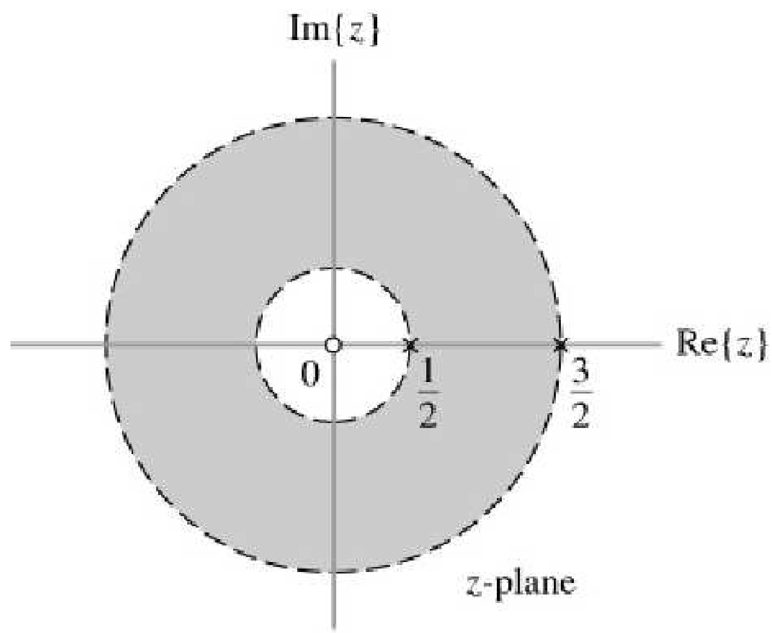


Figure 1.42: Pole-zero plot and ROC of $x[n]$

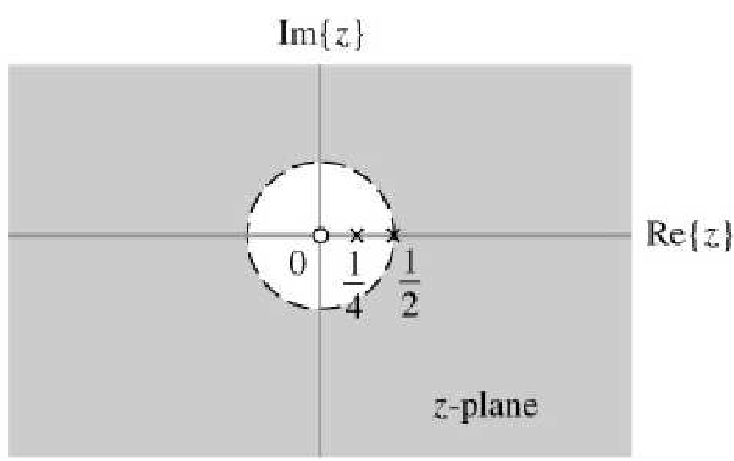


Figure 1.43: Pole-zero plot and ROC of $y[n]$

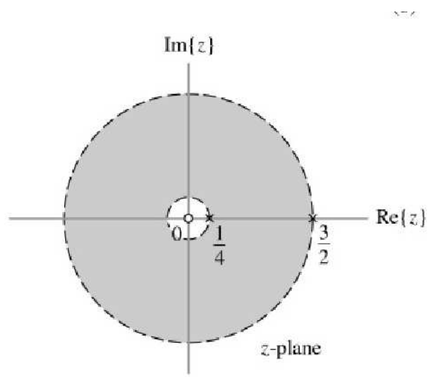


Figure 1.44: Pole-zero plot and ROC after cancellation

$$
\begin{aligned}
& a((\tfrac{1}{2})^n u[n] - (\tfrac{3}{2})^n u[-n-1] + \\
& \quad (\tfrac{1}{4})^n u[n] - (\tfrac{1}{2})^n u[n]) \\
& = a(-(\tfrac{3}{2})^n u[-n-1] + (\tfrac{1}{4})^n u[n])
\end{aligned}
$$

- The ROC is larger now, ie. $\frac{1}{4} < |z| < \frac{3}{2}$
- In the z-domain

$$
\begin{aligned}
aX(z) + bY(z) = & \\
& a\left(\frac{-z}{(z-\frac{1}{2})(z-\frac{3}{2})} + \frac{-\frac{1}{4}z}{(z-\frac{1}{4})(z-\frac{1}{2})}\right) \\
& = a\frac{-\frac{1}{4}z(z-\frac{3}{2}) - z(z-\frac{1}{4})}{(z-\frac{1}{4})(z-\frac{1}{2})(z-\frac{3}{2})}
\end{aligned}
$$

- In the z-domain

$$
aX(z) + bY(z) = a\frac{-\frac{5}{4}z(z-\frac{1}{2})}{(z-\frac{1}{4})(z-\frac{1}{2})(z-\frac{3}{2})}
$$

- The zero at $z = \frac{1}{2}$ cancels the pole at $z = \frac{1}{2}$, so

$$aX(z) + bY(z) = a\frac{-\frac{5}{4}z}{(z-\frac{1}{4})(z-\frac{3}{2})}$$

- The cancellation of the term $(\frac{1}{2})^n u[n]$ in time domain corresponds to cancellation of the pole at $z = \frac{1}{2}$ by a zero in the z-domain
- This pole defined the ROC boundary, so the ROC enlarges when the pole is canceled
- The ROC can be larger than the intersection if one or more terms in $x[n]$ or $y[n]$ cancel each other in the sum.

P2: Time reversal

- Time reversal or reflection corresponds to replacing z by z^{-1}. Hence, if R_x is of the form $a < |z| < b$ then the ROC of the reflected signal is $a < 1/|z| < b$ or $1/b < |z| < 1/a$

$$\text{If} \quad x[n] \stackrel{z}{\longleftrightarrow} X(z), \qquad \text{with ROC} \quad R_x$$

$$\text{Then} \quad x[-n] \stackrel{z}{\longleftrightarrow} X(\frac{1}{z}), \qquad \text{with ROC} \quad \frac{1}{R_x}$$

Proof: Time reversal

- Let $y[n] = x[-n]$
  $Y(z) = \sum_{n=-\infty}^{\infty} x[-n]z^{-n}$
  Let $l = -n$, then
  $Y(z) = \sum_{l=-\infty}^{\infty} x[l]z^{l}$

$Y(z) = \sum_{l=-\infty}^{\infty} x[l](\frac{1}{z})^{-l}$
$Y(z) = X(\frac{1}{z})$

P3: Time shift

- Time shift of n_o in the time domain corresponds to multiplication of z^{-n_o} in the z-domain

$$\text{If} \quad x[n] \overset{z}{\longleftrightarrow} X(z), \qquad \text{with ROC} \quad R_x$$

$$\text{Then} \quad x[n-n_o] \overset{z}{\longleftrightarrow} z^{-n_o}X(z),$$

$$\text{with ROC} \;\; R_x \;\; \text{except} \;\; z = 0 \text{ or } \; |z| = \infty$$

P3: Time shift, $n_o > 0$

- Multiplication by z^{-n_o} introduces a pole of order n_o at $z = 0$
- The ROC can not include $z = 0$, even if R_x does include $z = 0$
- If $X(z)$ has a zero of at least order n_o at $z = 0$ that cancels all of the new poles then ROC can include $z = 0$

P3: Time shift, $n_o < 0$

- Multiplication by z^{-n_o} introduces n_o poles at infinity
- If these poles are not canceled by zeros at infinity in $X(z)$ then the ROC of $z^{-n_o}X(z)$ can not include $|z| = \infty$

Proof: Time shift

- Let $y[n] = x[n-n_o]$
  $Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_o]z^{-n}$
  Let $l = n-n_o$, then
  $Y(z) = \sum_{l=-\infty}^{\infty} x[l]z^{-(l+n_o)}$
  $Y(z) = z^{-n_o}\sum_{l=-\infty}^{\infty} x[l]z^{-l}$
  $Y(z) = z^{-n_o}X(z)$

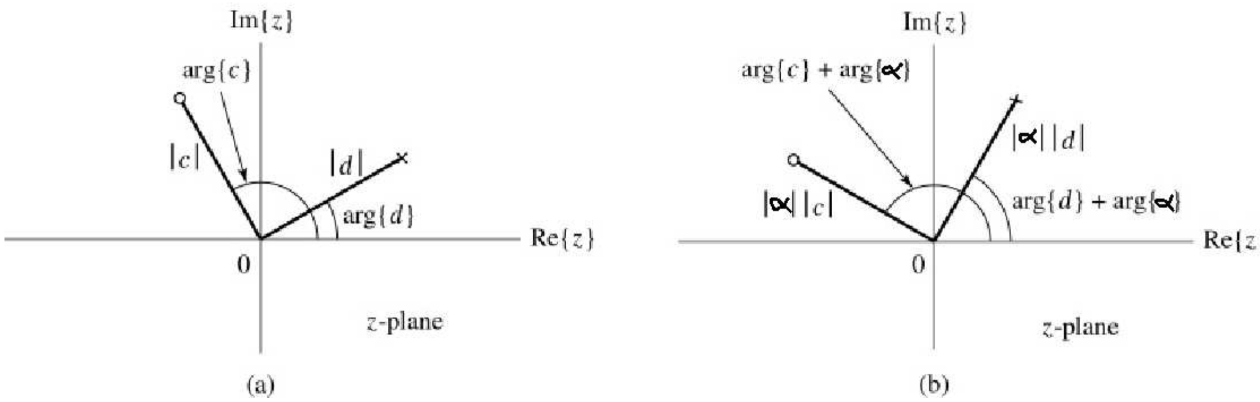
P4: Multiplication by α^n

- Let α be a complex number

$$\text{If} \quad x[n] \overset{z}{\longleftrightarrow} X(z), \qquad \text{with ROC} \quad R_x$$

$$\text{Then} \quad \alpha^n x[n] \overset{z}{\longleftrightarrow} X(\frac{z}{\alpha}), \qquad \text{with ROC} \quad |\alpha|R_x$$

- $|\alpha|R_x$ indicates that the ROC boundaries are multiplied by $|\alpha|$.
- If R_x is $a < |z| < b$ then the new ROC is $|\alpha|a < |z| < |\alpha|b$
- If $X(z)$ contains a pole d, ie. the factor $(z-d)$ is in the denominator then $X(\frac{z}{\alpha})$ has a factor $(z-\alpha d)$ in the denominator and thus a pole at αd.
- If $X(z)$ contains a zero c, then $X(\frac{z}{\alpha})$ has a zero at αc
- This indicates that the poles and zeros of $X(z)$ have their radii changed by $|\alpha|$
- Their angles are changed by $arg\{\alpha\}$



- If $|\alpha| = 1$ then the radius is unchanged and if α is +ve real number then the angle is unchanged

Proof: Multiplication by α^n

- Let $y[n] = \alpha^n x[n]$

$$Y(z) = \sum_{n=-\infty}^{\infty} \alpha^n x[n] z^{-n}$$

$$Y(z) = \sum_{l=-\infty}^{\infty} x[l] (\frac{z}{\alpha})^{-n}$$

$$Y(z) = X(\frac{z}{\alpha})$$

P5: Convolution

- Convolution in time domain corresponds to multiplication in the z-domain If $x[n] \overset{z}{\longleftrightarrow} X(z)$, with ROC R_x If $y[n] \overset{z}{\longleftrightarrow} Y(z)$, with ROC R_y Then $x[n] * y[n] \overset{z}{\longleftrightarrow} X(z)Y(z)$,

  with ROC at least $R_x \cap R_y$

- Similar to linearity the ROC may be larger than the intersection of R_x and R_y

Proof: Convolution

- Let $c[n] = x[n] * y[n]$

$$C(z) = \sum_{n=-\infty}^{\infty} (x[n] * y[n]) z^{-n}$$

$$C(z) = \sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} x[k] * y[n-k]) z^{-n}$$

$$C(z) = \sum_{k=-\infty}^{\infty} x[k] (\underbrace{(\sum_{n=-\infty}^{\infty} y[n-k]) z^{-(n-k)})}_{Y(z)} z^{-k}$$

$$C(z) = (\underbrace{\sum_{k=-\infty}^{\infty} x[k] z^{-k}}_{X(z)}) Y(z)$$

$$C(z) = X(z)Y(z)$$

P6: Differentiation in the z domain

- Multiplication by n in the time domain corresponds to differentiation with respect to z and multiplication of the result by $-z$ in the z-domain
  If $x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad$ with ROC R_x Then $nx[n] \stackrel{z}{\longleftrightarrow} -z\frac{d}{dz}X(z) \quad$ with ROC R_x
- ROC remains unchanged

Proof: Differentiation in the z domain

- We know

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Differentiate with respect to z

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n}z^{-1}$$

- Multiply with $-z$

$$-z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} -(-n)x[n]z^{-n}z^{-1}z$$

$$-z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

$$\text{Then } nx[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}X(z) \quad \text{with ROC } R_x$$

1.5.2 Examples

Example 1

Use the z-transform properties to determine the z-transform

- $x[n] = n((\frac{-1}{2})^n u[n]) * (\frac{1}{4})^{-n}u[-n]$

- Solution is:
  $a[n] = (\frac{-1}{2})^n u[n] \overset{z}{\longleftrightarrow} A(z) = \frac{1}{1+\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$
  $b[n] = na[n] \overset{z}{\longleftrightarrow} B(z) = -z\frac{d}{dz}A(z) = -z\frac{d}{dz}(\frac{1}{1+\frac{1}{2}z^{-1}}), \quad |z| > \frac{1}{2}$
  $b[n] = na[n] \overset{z}{\longleftrightarrow} B(z) = \frac{\frac{-1}{2}z}{(1+\frac{1}{2}z)^2}, \quad |z| > \frac{1}{2}$
  $c[n] = (\frac{1}{4})^n u[n] \overset{z}{\longleftrightarrow} C(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$

Use the z-transform properties to determine the z-transform

- $x[n] = n((\frac{-1}{2})^n u[n]) * (\frac{1}{4})^{-n}u[-n]$

- Solution continued

  $d[n] = c[-n] = (\frac{1}{4})^{-n}u[-n] \stackrel{z}{\longleftrightarrow} D(z) = C(\frac{1}{z}) = \frac{1}{1-\frac{1}{4}z}, \quad |z| < 4$

  $x[n] = (b[n] * d[n]) \stackrel{z}{\longleftrightarrow} X(z) = B(z)D(z), \quad \frac{1}{2} < |z| < 4$

  $x[n] = (b[n] * d[n]) \stackrel{z}{\longleftrightarrow} \frac{\frac{-1}{2}z}{(1+\frac{1}{2}z)^2}\frac{1}{(1-\frac{1}{4}z)}, \quad \frac{1}{2} < |z| < 4$

  $x[n] = (b[n] * d[n]) \stackrel{z}{\longleftrightarrow} \frac{2z}{(1+\frac{1}{2}z)^2(z-4)}, \quad \frac{1}{2} < |z| < 4$

Example 2

Use the z-transform properties to determine the z-transform

- $x[n] = a^n \cos(\Omega_o n)u[n]$, where a is real and +ve

- Solution is:

  $b[n] = a^n u[n] \stackrel{z}{\longleftrightarrow} B(z) = \frac{1}{1-az^{-1}}, \quad |z| > a$

  Put $\cos(\Omega_o n) = \frac{1}{2}e^{j\Omega_o n} + \frac{1}{2}e^{-j\Omega_o n}$, so we get

  $x[n] = \frac{1}{2}e^{j\Omega_o n}b[n] + \frac{1}{2}e^{-j\Omega_o n}b[n]$

Use the z-transform properties to determine the z-transform

- $x[n] = a^n \cos(\Omega_o n)u[n]$, where a is real and +ve

- Solution continued

  $x[n] \stackrel{z}{\longleftrightarrow} X(z) = \frac{1}{2}B(e^{j\Omega_o}z) + \frac{1}{2}B(e^{-j\Omega_o}z), \quad |z| > a$

  $x[n] \stackrel{z}{\longleftrightarrow} X(z) = \frac{1}{2}\frac{1}{1-ae^{j\Omega_o}z^{-1}} + \frac{1}{2}\frac{1}{1-ae^{-j\Omega_o}z^{-1}}, \quad |z| > a$

  $x[n] \stackrel{z}{\longleftrightarrow} X(z) = \frac{1}{2}(\frac{1-ae^{j\Omega_o}z^{-1}+1-ae^{-j\Omega_o}}{(1-ae^{j\Omega_o}z^{-1})(1-ae^{-j\Omega_o}z^{-1})})$

  $x[n] \stackrel{z}{\longleftrightarrow} X(z) = \frac{1-a\cos(\Omega_o)z^{-1}}{1-2a\cos(\Omega_o)z^{-1}+a^2z^{-2}}, \quad |z| > a$

1.5.3 Unsolved examples from [2]

Unsolved ex. 7.20(a)

Use the z-transform properties to determine the z-transform

- $x[n] = (\frac{1}{2})^n u[n] * 2^n u[-n-1]$
- Solution is:
  $a[n] = (\frac{1}{2})^n u[n] \overset{z}{\longleftrightarrow} A(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$ $b[n] = 2^n u[-n-1] \overset{z}{\longleftrightarrow} B(z) = \frac{1}{1-2z^{-1}}, \quad |z| < 2$ $a[n] * b[n] \overset{z}{\longleftrightarrow} X(z) = A(z)B(z), \quad \frac{1}{2} < |z| < 2$
  $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \frac{1}{1-2z^{-1}}, \quad \frac{1}{2} < |z| < 2$

Unsolved ex. 7.20(b)

Use the z-transform properties to determine the z-transform

- $x[n] = n((\frac{1}{2})^n u[n] * (\frac{1}{4})^n u[n-2])$
- Solution is:
  $a[n] = (\frac{1}{2})^n u[n] \overset{z}{\longleftrightarrow} A(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$ $b[n] = (\frac{1}{4})^n u[n] \overset{z}{\longleftrightarrow} B(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$ $c[n] = b[n-2] \overset{z}{\longleftrightarrow} C(z) = \frac{z^{-2}}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$ $x[n] = n(a[n] * c[n]) \overset{z}{\longleftrightarrow} X(z) = -z\frac{d}{dz}A(z)C(z) \quad |z| > \frac{1}{2}$

1.5.4 Exercises

1. Solve the difference equation $x_{k+2} - 4x_{k+1} - 5x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.
2. Solve the difference equation $x_{k+2} - 9x_{k+1} + 20x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.
3. Solve the difference equation $x_{k+2} + 5x_{k+1} + 6x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.
4. Solve the difference equation $x_{k+2} + 2x_{k+1} - 48x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.
5. Solve the difference equation $x_{k+2} + 7x_{k+1} - 18x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.
6. Solve the difference equation $x_{k+2} - 6x_{k+1} + 5x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.

Answers

1. So, take the z-transform of both sides

$$z^2X - z - 4zX - 5X = 0 \tag{1.21}$$

hence

$$X = \frac{z}{z^2 - 4z - 5} \tag{1.22}$$

Move the z to the left and do partial fractions,

$$\frac{1}{z}X = \frac{1}{z^2 - 4z - 5} = \frac{1}{(z-5)(z+1)} = \frac{1}{6(z-5)} - \frac{1}{6(z+1)} \tag{1.23}$$

Thus

$$X = \frac{z}{6(z-5)} - \frac{z}{6(z+1)} \tag{1.24}$$

and

$$x_k = \frac{1}{6}5^k - \frac{1}{6}(-1)^k \tag{1.25}$$

2. So, take the z-transform of both sides

$$z^2X - z - 9zX + 20X = 0 \tag{1.26}$$

hence

$$X = \frac{z}{z^2 - 9z + 20} \tag{1.27}$$

Move the z to the left and do partial fractions,

$$\frac{1}{z}X = \frac{1}{z^2 - 9z + 20} = \frac{1}{(z-5)(z-4)} = \frac{1}{z-5} - \frac{1}{z-4} \tag{1.28}$$

Thus

$$X = \frac{z}{z-5} - \frac{z}{z-4} \tag{1.29}$$

and

$$x_k = 5^k - 4^k \tag{1.30}$$

3. So, take the z-transform of both sides

$$z^2X - z + 5zX + 6X = 0 \tag{1.31}$$

hence

$$X = \frac{z}{z^2+5z+6} \tag{1.32}$$

Move the z to the left and do partial fractions,

$$\frac{1}{z}X = \frac{1}{z^2+5z+6} = \frac{1}{(z+2)(z+3)} = \frac{1}{z+2} - \frac{1}{z+3} \tag{1.33}$$

Thus

$$X = \frac{z}{z+2} - \frac{z}{z+3} \tag{1.34}$$

and

$$x_k = (-2)^k - (-3)^k \tag{1.35}$$

4. So, take the z-transform of both sides

$$z^2X - z + 2zX - 48X = 0 \tag{1.36}$$

hence

$$X = \frac{z}{z^2+2z-48} \tag{1.37}$$

Move the z to the left and do partial fractions,

$$\frac{1}{z}X = \frac{1}{z^2+2z-48} = \frac{1}{(z+8)(z-6)} = -\frac{1}{14(z+8)} + \frac{1}{14(z-6)} \tag{1.38}$$

Thus

$$X = -\frac{z}{14(z+8)} + \frac{z}{14(z-6)} \tag{1.39}$$

and

$$x_k = -\frac{1}{14}(-8)^k + \frac{1}{14}6^k \tag{1.40}$$

5. So, take the z-transform of both sides

$$z^2X - z + 7zX - 18X = 0 \tag{1.41}$$

hence

$$X = \frac{z}{z^2+7z-18} \tag{1.42}$$

Move the z to the left and do partial fractions,

$$\frac{1}{z}X = \frac{1}{z^2+7z-18} = \frac{1}{(z-2)(z+9)} = \frac{1}{11(z-2)} - \frac{1}{11(z+9)} \tag{1.43}$$

Thus

$$X = \frac{z}{11(z-2)} - \frac{z}{11(z+9)} \tag{1.44}$$

and

$$x_k = \frac{1}{11}(-2)^k - \frac{1}{11}(-9)^k \tag{1.45}$$

6. So, take the z-transform of both sides

$$z^2X - z - 6zX + 5X = 0 \tag{1.46}$$

hence

$$X = \frac{z}{z^2 - 6z + 5} \tag{1.47}$$

Move the z to the left and do partial fractions,

$$\frac{1}{z}X = \frac{1}{z^2 - 6z + 5} = \frac{1}{(z-5)(z-1)} = \frac{1}{4(z-5)} - \frac{1}{4(z-1)} \tag{1.48}$$

Thus

$$X = \frac{z}{4(z-5)} - \frac{z}{4(z-1)} \tag{1.49}$$

and

$$x_k = \frac{1}{4}5^k - \frac{1}{4} \tag{1.50}$$

Conclusions

- Properties of z-transform
  - Linearity
  - Time reversal
  - Time shift
  - Multiplication by α^n
  - Convolution
  - Differentiation in the z-domain

1.6 Class 6: Inverse z-transform

Outline of today's class

- Inverse z-transform
- Directly from inverse z-transform equation requires understanding of complex variable theory
- Alternate methods of inverse z-transform
  - Partial fraction method: uses basic z-transform pairs and properties
  - Power series method: express $X(z)$ in-terms of z^{-1} and find by inspection

1.6.1 Partial fraction method

- In case of LTI systems, commonly encountered form of z-transform is

$$X(z) = \frac{B(z)}{A(z)}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

  Usually $M < N$

- If $M > N$ then use long division method and express $X(z)$ in the form

$$X(z) = \sum_{k=0}^{M-N} f_k z^{-k} + \frac{\tilde{B}(z)}{A(z)}$$

where $\tilde{B}(z)$ now has the order one less than the denominator polynomial and use partial fraction method to find z-transform

- The inverse z-transform of the terms in the summation are obtained from the transform pair and time shift property

$$1 \stackrel{z}{\longleftrightarrow} \delta[n]$$

$$z^{-n_o} \stackrel{z}{\longleftrightarrow} \delta[n-n_o]$$

- If $X(z)$ is expressed as ratio of polynomials in z instead of z^{-1} then convert into the polynomial of z^{-1}

- Convert the denominator into product of first-order terms

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

where d_k are the poles of $X(z)$

For distinct poles

- For all distinct poles, the $X(z)$ can be written as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})}$$

- Depending on ROC, the inverse z-transform associated with each term is then determined by using the appropriate transform pair

- We get

$$A_k (d_k)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{A_k}{1 - d_k z^{-1}},$$

$$\text{with ROC} \quad z > d_k \qquad \text{OR}$$

$$-A_k(d_k)^n u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{A_k}{1-d_k z^{-1}},$$

$$\text{with ROC} \quad z < d_k$$

- For each term the relationship between the ROC associated with $X(z)$ and each pole determines whether the right-sided or left sided inverse transform is selected

For Repeated poles

- If pole d_i is repeated r times, then there are r terms in the partial-fraction expansion associated with that pole

$$\frac{A_{i_1}}{1-d_i z^{-1}}, \frac{A_{i_2}}{(1-d_i z^{-1})^2}, \ldots, \frac{A_{i_r}}{(1-d_i z^{-1})^r}$$

- Here also, the ROC of $X(z)$ determines whether the right or left sided inverse transform is chosen.

$$A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{A}{(1-d_i z^{-1})^m}, \qquad \text{with ROC}|z| > d_i$$

- If the ROC is of the form $|z| < d_i$, the left-sided inverse z-transform is chosen, ie.

$$-A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{A}{(1-d_i z^{-1})^m}, \quad \text{with ROC}|z| < d_i$$

Deciding ROC

- The ROC of $X(z)$ is the intersection of the ROCs associated with the individual terms in the partial fraction expansion.

- In order to chose the correct inverse z-transform, we must infer the ROC of each term from the ROC of $X(z)$.
- By comparing the location of each pole with the ROC of $X(z)$.
- Chose the right sided inverse transform: if the ROC of $X(z)$ has the radius greater than that of the pole associated with the given term
- Chose the left sided inverse transform: if the ROC of $X(z)$ has the radius less than that of the pole associated with the given term

1.6.2 Examples

Example 1a

Example of proper rational function

- Find the inverse z-transform of

$$X(z) = \frac{1 - z^{-1} + z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})(1 - z^{-1})},$$

with ROC $1 < |z| < 2$

- Solution: Use partial fraction and rewrite the expression

$$X(z) = \frac{A_1}{(1 - \frac{1}{2}z^{-1})} + \frac{A_2}{(1 - 2z^{-1})} + \frac{A_3}{(1 - z^{-1})}$$

- Solving for A_1, A_2 and A_3 gives the values as $A_1 = 1$, $A_2 = 2$ and $A_3 = -2$,

$$X(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})} + \frac{2}{(1 - 2z^{-1})} - \frac{2}{(1 - z^{-1})}$$

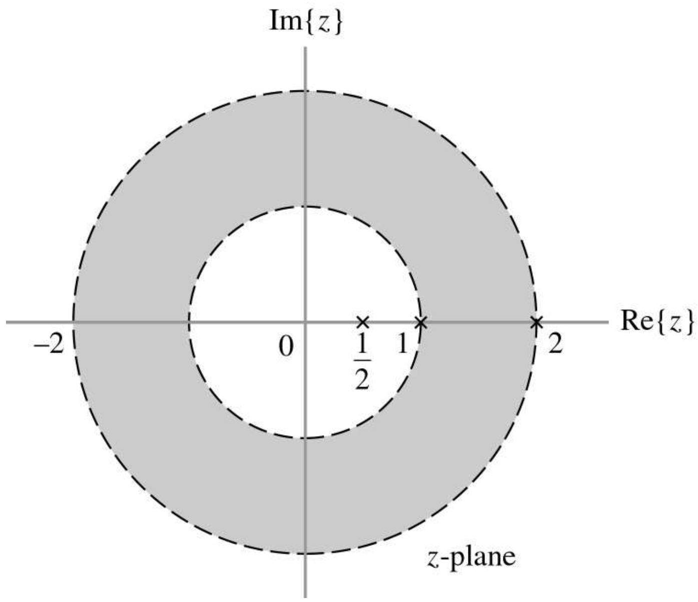


Figure 1.45: ROC for Example 1

- Find the inverse z-transform of the individual terms
- Use the relationship between the location of poles and the ROC of $X(z)$

- From figure, one pole is at $z = \frac{1}{2}$, ROC has a radius greater than the pole at $z = \frac{1}{2}$. This term corresponds to right-sided sequence.

$$(\frac{1}{2})^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}$$

- Another pole is at $z = 2$, ROC has a radius smaller than the pole at $z = 2$. This term corresponds to left-sided sequence.

$$-2(2)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{2}{1-2z^{-1}}$$

- Third pole is at $z = 1$, ROC has a radius greater than the pole at $z = 1$. This term corresponds to right-sided sequence.

$$-2u[n] \overset{z}{\longleftrightarrow} -\frac{2}{1-z^{-1}}$$

- Combining the individual terms gives

$$x[n] = (\frac{1}{2})^n u[n] - 2(2)^n u[-n-1] - 2u[n]$$

- (b) Repeat the example 1 with the ROC $\frac{1}{2} < |z| < 1$

Example 1b

- From figure, one pole is at $z = \frac{1}{2}$, ROC has a radius greater than the pole at $z = \frac{1}{2}$. This term corresponds to right-sided sequence.

$$(\frac{1}{2})^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}$$

- Another pole is at $z = 2$, ROC has a radius smaller than the pole at $z = 2$. This term corresponds to left-sided sequence.

$$-2(2)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{2}{1-2z^{-1}}$$

- Third pole is at $z = 1$, ROC has a radius smaller than the pole at $z = 1$. This term corresponds to left-sided sequence.

$$2u[-n-1] \overset{z}{\longleftrightarrow} -\frac{2}{1-z^{-1}}$$

- Combining the individual terms gives

$$x[n] = (\frac{1}{2})^n u[n] - 2(2)^n u[-n-1] + 2u[-n-1]$$

- (c) Repeat the example 1 with the ROC $|z| < \frac{1}{2}$

Example 1c

- From figure, one pole is at $z = \frac{1}{2}$, ROC has a radius smaller than the pole at $z = \frac{1}{2}$. This term corresponds to left-sided sequence.

$$-(\frac{1}{2})^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}z^{-1}}$$

- Another pole is at $z = 2$, ROC has a radius smaller than the pole at $z = 2$. This term corresponds to left-sided sequence.

$$-2(2)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{2}{1 - 2z^{-1}}$$

- Third pole is at $z = 1$, ROC has a radius smaller than the pole at $z = 1$. This term corresponds to left-sided sequence.

$$2u[-n-1] \overset{z}{\longleftrightarrow} -\frac{2}{1 - z^{-1}}$$

- Combining the individual terms gives

$$x[n] = -(\frac{1}{2})^n u[-n-1] - 2(2)^n u[-n-1]$$
$$+ 2u[-n-1]$$

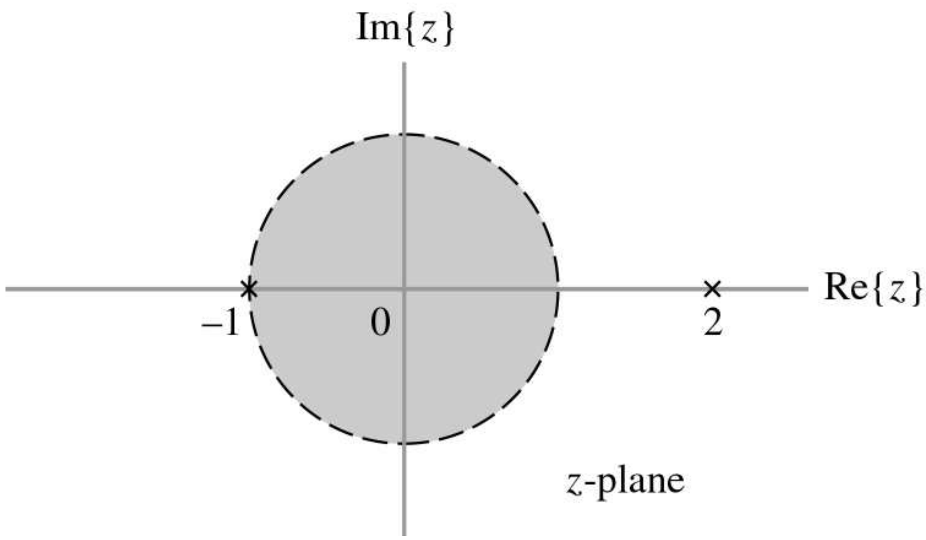


Figure 1.46: ROC for Example 2

Example 2

Example of improper rational function

- Find the inverse z-transform of

$$X(z) = \frac{z^3 - 10z^2 - 4z + 4}{2z^2 - 2z - 4},$$

with ROC $|z| < 1$

- Solution: Find the locations of poles by determining the roots of denominator polynomial

$$2z^2 - 4z + 4 = 0$$

- We get the poles at $z = -1$ and $z = 2$

- The ROC and pole locations are

- Convert $X(z)$ into a ratio of polynomials in z^{-1}

$$X(z) = \frac{1}{2}z(\frac{1-10z^{-1}-4z^{-2}+4z^{-3}}{1-z^{-1}-2z^{-2}}) = \frac{1}{2}zA(z)$$

- The factor $\frac{1}{2}z$ can be incorporated using time-shift property
- Use long division method and reduce the order of the numerator polynomial
- We get quotient as $-2z^{-1}+3$ and remainder as $-5z^{-1}-2$, then

$$A(z) = -2z^{-1}+3+\frac{-5z^{-1}-2}{1-z^{-1}-2z^{-2}}$$

- We have $X(z) = \frac{1}{2}zA(z)$ and $A(z) = -2z^{-1}+3+W(z)$
- Use partial fraction expansion to find the inverse z-transform of $W(z)$

$$W(z) = \frac{-5z^{-1}-2}{1-z^{-1}-2z^{-2}} = \frac{1}{1+z^{-1}} - \frac{3}{1-2z^{-1}}$$

- We have $X(z) = \frac{1}{2}zA(z)$ and $A(z) = -2z^{-1}+3+W(z)$
- So we can write

$$A(z) = -2z^{-1}+3+\frac{1}{1+z^{-1}} - \frac{3}{1-2z^{-1}}$$

with ROC$|z| < 1$

- The ROC has a smaller radius than either pole, hence the inverse z-transform of $A(z)$

$$a[n] = -2\delta[n-1] + 3\delta[n] - (-1)^n u[-n-1]$$
$$+3(2)^n u[-n-1]$$

- Apply the time shift property

$$x[n-n_o] \stackrel{z}{\longleftrightarrow} z^{-n_o} X(z),$$
$$\text{with ROC } \; R_x \; \text{ except } \; z = 0 \text{ or } \; |z| = \infty$$

- Here, we have $X(z) = \frac{1}{2} z A(z)$, so $n_o = -1$ and

$$x[n] = \frac{1}{2} a[n+1]$$

- So we get $x[n]$ as

$$x[n] = \frac{1}{2} a[n+1] = -\delta[n] + \frac{3}{2}\delta[n+1] - \frac{1}{2}(-1)^{n+1}$$
$$u[-n-2] + 3(2)^n u[-n-2]$$

1.6.3 Unsolved example from [2]

Unsolved ex. 7.24(a)

Example of proper rational function

- Find the inverse z-transform of

$$X(z) = \frac{1+\frac{7}{6}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}$$

with ROC $|z| > \frac{1}{2}$

- We can write

$$X(z) = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{3}z^{-1}}$$

$$\text{Put} \quad z = 0 \quad \text{and we get} \quad 1 = A + B$$

$$\text{Put} \quad z = 1 \quad \text{and we get} \quad \frac{13}{6} = \frac{4}{3}A + \frac{1}{2}B$$

- Solve for A and B, we get $A = 2$ and $B = -1$
- Put the vales of A and B we get

$$X(z) = \frac{2}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1+\frac{1}{3}z^{-1}}$$

- Take inverse z-transform of $X(z)$, and $x[n]$ is right-sided sequence

$$x[n] = 2(\frac{1}{2})^n u[n] - (-\frac{1}{3})^n u[n]$$

Unsolved ex. 7.24(b)

Example of proper rational function

- Find the inverse z-transform of

$$X(z) = \frac{1+\frac{7}{6}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}$$

with ROC $|z| < \frac{1}{3}$

- We can write

$$X(z) = \frac{2}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1+\frac{1}{3}z^{-1}}$$

- Take inverse z-transform of $X(z)$, and $x[n]$ is left-sided sequence

$$x[n] = -2(\frac{1}{2})^n u[-n-1] + (-\frac{1}{3})^n u[-n-1]$$

Unsolved ex. 7.24(c)

Example of proper rational function

- Find the inverse z-transform of

$$X(z) = \frac{1+\frac{7}{6}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}$$

with ROC $\frac{1}{3} < |z| < \frac{1}{2}$

- We can write

$$X(z) = \frac{2}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1+\frac{1}{3}z^{-1}}$$

- Take inverse z-transform of $X(z)$, and $x[n]$ is two-sided sequence

$$x[n] = -2(\frac{1}{2})^n u[-n-1] - (-\frac{1}{3})^n u[n]$$

1.6.4 Exercises

1. Solve the difference equation $x_{k+2} - 4x_{k+1} - 5x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.

2. Solve the difference equation $x_{k+2} - 9x_{k+1} + 20x_k = 2^k$ with $x_0 = 0$ and $x_1 = 0$.

3. Solve the difference equation $x_{k+2} + 5x_{k+1} + 6x_k = (-2)^k$ with $x_0 = 0$ and $x_1 = 0$.

4. Solve the difference equation $x_{k+2} + 2x_{k+1} - 48x_k = 0$ with $x_0 = 4$ and $x_1 = 2$.

5. Solve the difference equation $x_{k+2} + 7x_{k+1} - 18x_k = \delta_k$ with $x_0 = 0$ and $x_1 = 0$.

Answers

1. So take the z-transform of both sides

$$z^2X - z - 4zX - 5X = 0 \tag{1.51}$$

and move things around to get X/z on one side and then do partial fractions

$$\frac{1}{z}X = \frac{1}{z^2 - 4z - 5} = \frac{1}{(z-5)(z+1)} = \frac{A}{z-5} + \frac{B}{z+1} \tag{1.52}$$

In the usual way, we have

$$1 = A(z+1) + B(z-5) \tag{1.53}$$

and putting $z = 5$ gives $A = 1/6$ and putting $z = -1$ gives $B = -1/6$. Now

$$X = \frac{z}{6(z-5)} - \frac{1}{6(z+1)} \tag{1.54}$$

and hence

$$x_k = \frac{1}{6}5^k - \frac{1}{6}(-1)^k \tag{1.55}$$

2. So, in this example, the right hand side of the difference equation is not zero. Taking the z-transform of both sides we get

$$z^2X - 9zX + 20X = \mathcal{Z}[(2^k)] = \frac{z}{z-2} \tag{1.56}$$

Hence, since $z^2 - 9z + 20 = (z-5)(z-4)$

$$\frac{1}{z}X = \frac{1}{(z-5)(z-4)(z-2)} \tag{1.57}$$

The usual partial fractions tells us that

$$\frac{1}{(z-5)(z-4)(z-2)} = \frac{1}{3(z-5)} - \frac{1}{2(z-4)} + \frac{1}{6(z-2)} \tag{1.58}$$

and so

$$x_k = \frac{1}{3}5^k - \frac{1}{2}4^k + \frac{1}{6}2^k \tag{1.59}$$

3. Again, taking the z-transform of both sides we have

$$z^2X + 5zX + 6X = \frac{z}{z+2} \tag{1.60}$$

Now, since $z^2 + 5z + 6 = (z+2)(z+3)$

$$\frac{1}{z}X = \frac{1}{(z+2)^2(z+3)} \tag{1.61}$$

and there is a repeated root. The partial fraction expansion with a repeated root includes the root and its square, so we get

$$\frac{1}{(z+2)^2(z+3)} = \frac{A}{z+2} + \frac{B}{(z+2)^2} + \frac{C}{z+3} \tag{1.62}$$

and so

$$1 = A(z+2)(z+3) + B(z+3) + C(z+2)^2 \tag{1.63}$$

Choosing $z = -2$ gives $B = 1$ and $z = -3$ gives $C = 1$. No value of z will give A on its own, so we choose another convenient value and put in the known values of B and C:

$$1 = 6A + 3 + 4 \tag{1.64}$$

so $A = -1$. Now, this means

$$X = -\frac{z}{z+2} + \frac{z}{(z+2)^2} + \frac{z}{z+3} \tag{1.65}$$

and so

$$x_k = (-2)^k + k(-2)^{k-1} + (-3)^k \tag{1.66}$$

4. Take the z-transform of both sides, taking care to note the initial conditions

$$z^2X - 4z^2 - 2z + 2(zX - 4z) - 48X = 0 \tag{1.67}$$

Thus

$$z^2X + 2zX - 48X = 4z^2 - 10z \tag{1.68}$$

giving

$$\frac{1}{z}X = \frac{4z-10}{(z+8)(z-6)} = \frac{A}{z+8} + \frac{B}{z-6} \tag{1.69}$$

Multiplying across we get

$$4z - 10 = A(z-6) + B(z+8) \tag{1.70}$$

Choosing $z = -8$ we have

$$-42 = -14A \tag{1.71}$$

implying $A = 3$. Choosing $z = 6$

$$14 = 14B \tag{1.72}$$

so $B = 1$ and we get

$$X = \frac{3z}{z+8} + \frac{z}{z-6} \tag{1.73}$$

and

$$x_k = 3(-8)^k + 6^k \tag{1.74}$$

5. Now, taking the z-transform and using $\mathcal{Z}[(\delta_k)] = 1$

$$z^2X + 7zX - 18X = 1 \tag{1.75}$$

and so

$$X = \frac{1}{z^2+7z-18} = \frac{1}{(z-9)(z+2)} = \frac{1}{11(z-9)} - \frac{1}{11(z+2)} \tag{1.76}$$

Thus

$$X = \frac{1}{z}\left(\frac{z}{11(z-9)} - \frac{z}{11(z+2)}\right) \tag{1.77}$$

and so, using the delay theorem, we have

$$x_k = \begin{cases} 0 & k = 0 \\ \frac{1}{11}9^{k-1} - \frac{1}{11}(-2)^{k-1} & k > 0 \end{cases} \tag{1.78}$$

Partial fraction method

- It can be applied to complex valued poles
- Generally the expansion coefficients are complex valued
- If the coefficients in $X(z)$ are real valued, then the expansion coefficients corresponding to complex conjugate poles will be complex conjugate of each other

- Here we use information other than ROC to get unique inverse transform
- We can use causality, stability and existence of DTFT
- If the signal is known to be causal then right sided inverse transform is chosen
- If the signal is stable, then t is absolutely summable and has DTFT
- Stability is equivalent to existence of DTFT, the ROC includes the unit circle in the z-plane, ie. $|z| = 1$
- The inverse z-transform is determined by comparing the poles and the unit circle
- If the pole is inside the unit circle then the right-sided inverse z-transform is chosen
- If the pole is outside the unit circle then the left-sided inverse z-transform is chosen

Conclusions

- Inverse z-transform
- Directly from inverse z-transform equation requires understanding of complex variable theory
- Alternate methods of inverse z-transform

- Partial fraction method: uses basic z-transform pairs and properties

1.7 Class 7: Inverse z-transform

Outline of today's class

- Power series method of finding inverse z-transform: express $X(z)$ in-terms of z^{-1} and find the inverse z-transform by inspection
- The transfer function: another method of describing the system and also provides a method to find inverse z-transform

Partial fraction method

- If $X(z)$ is expressed as ratio of polynomials in z instead of z^{-1} then convert into the polynomial of z^{-1}
- If $X(z)$ is in improper fraction the convert into proper fraction and use time shift property
- Convert the denominator into product of first-order terms
- Depending on ROC, the inverse z-transform associated with each term is then determined by using the appropriate transform pair for both distinct and real poles
- We can use causality, stability and existence of DTFT
- If the signal is stable, then it is absolutely summable and has DTFT

1.7.1 Power series expansion

- Express $X(z)$ as a power series in z^{-1} or z as given in z-transform equation

- The values of the signal $x[n]$ are then given by coefficient associated with z^{-n}
- Main disadvantage: limited to one sided signals
- Signals with ROCs of the form $|z| > a$ or $|z| < a$
- If the ROC is $|z| > a$, then express $X(z)$ as a power series in z^{-1} and we get right sided signal
- If the ROC is $|z| < a$, then express $X(z)$ as a power series in z and we get left sided signal

Using long division

- Find the z-transform of

$$X(z) = \frac{2 + z^{-1}}{1 - \frac{1}{2}z^{-1}}, \text{with ROC} \quad |z| > \frac{1}{2}$$

- Solution is: use long division method to write $X(z)$ as a power series in z^{-1}, since ROC indicates that $x[n]$ is right sided sequence
- We get

$$X(z) = 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \ldots$$

- Compare with z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- We get

$$x[n] = 2\delta[n] + 2\delta[n-1] + \delta[n-2] + \frac{1}{2}\delta[n-3] + \ldots$$

- If we change the ROC to $|z| < \frac{1}{2}$, then expand $X(z)$ as a power series in z using long division method

- We get

$$X(z) = -2 - 8z - 16z^2 - 32z^3 + \ldots$$

- We can write $x[n]$ as

$$x[n] = -2\delta[n] - 8\delta[n+1] - 16\delta[n+2] - 32\delta[n+3] + \ldots$$

Power series expansion

- Find the z-transform of

$$X(z) = e^{z^2}, \text{with ROC all } z \text{ except } |z| = \infty$$

- Solution is: use power series expansion for e^a and is given by

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$$

- We can write $X(z)$ as

$$X(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!}$$

$$X(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!}$$

- We can write $x[n]$ as

$$x[n] = \begin{cases} 0 & n > 0 \text{ or } n \text{ is odd} \\ \frac{1}{(\frac{-n}{2})!}, & \text{otherwise} \end{cases}$$

1.7.2 Unsolved examples from [2]

Unsolved ex.7.28(a)

- Find $x[n]$ using power series method if

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-2}}, \quad |z| > \frac{1}{4}$$

- Write $X(z)$ in the power series

$$X(z) = \sum_{k=0}^{\infty} (\frac{1}{4}z^{-2})^k$$

- Write $x[n]$ by comparing with $X(z)$

$$x[n] = \sum_{k=0}^{\infty} (\frac{1}{4})^k \delta[n-2k]$$

$$x[n] = \begin{cases} (\frac{1}{4})^{\frac{n}{2}}, & n \text{ even and } n \geq 0 \\ 0 & n \text{ odd} \end{cases}$$

Unsolved ex.7.28(b)

- Find $x[n]$ using power series method if

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-2}}, \quad |z| < \frac{1}{4}$$

- Write $X(z)$ in the power series

$$X(z) = \frac{-4z^2}{1-4z^2} = -4z^2\sum_{k=0}^{\infty}(4z^2)^k$$

- Write $X(z)$ in the power series

$$X(z) = -4z^2\sum_{k=0}^{\infty}(2z)^{2k} = -\sum_{k=0}^{\infty}2^{2(k+1)}z^{2(k+1)}$$

- Write $x[n]$ by comparing with $X(z)$

$$x[n] = -\sum_{k=0}^{\infty}2^{2(k+1)}\delta[n+2(k+1)]$$

Unsolved ex.7.28(d)

- Find $x[n]$ using power series method if

$$X(z) = ln(1+z^{-1}), \quad |z| > 0$$

- We know

$$ln(1+a) = \sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{k}(a)^k$$

- Write $X(z)$ in the power series

$$X(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k} (z^{-1})^k$$

- Write $x[n]$ by comparing with $X(z)$

$$x[n] = -\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k} \delta[n-k]$$

1.7.3 The transfer function

- We have defined the transfer function as the z-transform of the impulse response of an LTI system

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- Then we have $y[n] = x[n] * h[n]$ and $Y(z) = X(z)H(z)$
- This is another method of representing the system
- The transfer function can be written as

$$H(z) = \frac{Y(z)}{X(z)}$$

- This is true for all z in the ROCs of $X(z)$ and $Y(z)$ for which $X(z)$ in nonzero
- The impulse response is the z-transform of the transfer function
- We need to know ROC in order to uniquely find the impulse response

- If ROC is unknown, then we must know other characteristics such as stability or causality in order to uniquely find the impulse response

System identification

- Finding a system description by using input and output is known as system identification
- Ex1: find the system, if the input is $x[n] = (-1//3)^n u[n]$ and the out is $y[n] = 3(-1)^n u[n] + (1/3)^n u[n]$
- Solution: Find the z-transform of input and output. Use $X(z)$ and $Y(z)$ to find $H(z)$, then find $h(n)$ using the inverse z-transform

$$X(z) = \frac{1}{(1+(\frac{1}{3})z^{-1})}, \qquad \text{with ROC } |z| > \frac{1}{3}$$

$$Y(z) = \frac{3}{(1+z^{-1})} + \frac{1}{(1-(\frac{1}{3})z^{-1})}, \quad \text{with ROC } |z| > 1$$

- We can write $Y(z)$ as

$$Y(z) = \frac{4}{(1+z^{-1})(1-(\frac{1}{3})z^{-1})}, \quad \text{with ROC } |z| > 1$$

- We know $H(z) = Y(z)/X(z)$, so we get

$$H(z) = \frac{4(1+(\frac{1}{3})z^{-1})}{(1+z^{-1})(1-(\frac{1}{3})z^{-1})} \quad \text{with ROC } |z| > 1$$

- We need to find inverse z-transform to find $x[n]$, so use partial fraction and write $H(z)$ as

$$H(z) = \frac{2}{1+z^{-1}} + \frac{2}{1-(\frac{1}{3})z^{-1}} \quad \text{with ROC } \ |z| > 1$$

- Impulse response $x[n]$ is given by

$$h[n] = 2(-1)^n u[n] + 2(1/3)^n u[n]$$

- Ex2: If the impulse response of an LTI system is $h[n] = (1/2)^n u[n]$. Find the input if the out is $y[n] = (1/2)^n u[n] + (-1/2)^n u[n]$
- Find $H(z)$ and $Y(z)$, we have $X(z) = Y(z)/H(z)$, find $x[n]$ by taking the inverse z-transform
- We get $x[n]$ as $x[n] = 2(-1/2)^n u[n]$

Relation between transfer function and difference equation

- The transfer can be obtained directly from the difference-equation description of an LTI system
- We know that

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- We know that the transfer function $H(z)$ is an eigen value of the system associated with the eigen function z^n, ie. if $x[n] = z^n$ then the output of an LTI system $y[n] = z^n H(z)$
- Put $x[n-k] = z^{n-k}$ and $y[n-k] = z^{n-k}H(z)$ in the difference equation,

we get

$$z^n \sum_{k=0}^{N} a_k z^{-k} H(z) = z^n \sum_{k=0}^{M} b_k z^{-k}$$

- We can solve for $H(z)$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The transfer function described by a difference equation is a ratio of polynomials in z^{-1} and is termed as a rational transfer function.
- The coefficient of z^{-k} in the numerator polynomial is the coefficient associated with $x[n-k]$ in the difference equation
- The coefficient of z^{-k} in the denominator polynomial is the coefficient associated with $y[n-k]$ in the difference equation
- This relation allows us to find the transfer function and also find the difference equation description for a system, given a rational function

Example 1

- Find the transfer function and the impulse response for the causal LTI system described by

$$y[n] - \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = -x[n] + 2x[n-1]$$

- We can write

$$H(z) = \frac{-1 + 2z^{-1}}{1 - (\frac{1}{4})z^{-1} - \frac{3}{8}z^{-2}}$$

- Write $H(z)$ in terms of partial fraction expansion

$$H(z) = \frac{2}{1+\frac{1}{2}z^{-1}} + \frac{1}{1-(\frac{3}{4})z^{-1}}$$

- This is causal system so we can write

$$h[n] = -2(-\frac{1}{2})^n u[n] + (\frac{3}{4})^n u[n]$$

Example 2

- Find the the difference equation description of an LTI system with transfer function

$$H(z) = \frac{5z+2}{z^2+3z+2}$$

- Solution: Rewrite $H(z)$ as a ratio of polynomials in z^{-1}

$$H(z) = \frac{5z^{-1}+2z^{-2}}{1+3z^{-1}+2z^{-2}}$$

- Compare this with the difference equation description of the transfer function $H(z)$, we get $M=2, N=2, b_0=0, b_1=5$, $b_2=2, a_0=1, a_1=3$ and $a_2=2$

- We can write the difference equation as

$$y[n]+3y[n-1]+2y[n-2] = 5x[n-1]+2x[n-2]$$

Transfer function

- The poles and zeros of a rational function offer much insight into LTI system characteristics

- The transfer function can be expressed in pole-zero form by factoring the numerator and denominator polynomial
- If c_k and d_k are zeros and poles of the system respectively and $\tilde{b} = b_0/a_0$ is the gain factor, then

$$H(z) = \frac{\tilde{b}\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- This form assumes there are no poles and zeros at $z = 0$
- The p^{th} order pole at $z = 0$ occurs when $b_0 = b_1 = \ldots = b_{p-1} = 0$
- The l^{th} order zero at $z = 0$ occurs when $a_0 = a_1 = \ldots = a_{l-1} = 0$
- Then we can write $H(z)$ as

$$H(z) = \frac{\tilde{b}z^{-p}\prod_{k=1}^{M-p}(1 - c_k z^{-1})}{z^{-l}\prod_{k=1}^{N-l}(1 - d_k z^{-1})}$$

  where $\tilde{b} = b_p/a_l$

- In the example we had first order pole at $z = 0$
- The poles, zeros and gain factor $\tilde{b}$ uniquely determine the transfer function
- This is another description for input-output behavior of the system
- The poles are the roots of characteristic equation

Conclusions

- Power series method of finding inverse z-transform
- The transfer function

1.8 Class 8: Causality, stability and Inverse systems

Outline of today's class

- Causality of an LTI system and the inverse z-transform
- Stability of an LTI system and the inverse z-transform
- Inverse systems
- Stability and causality of an inverse system

1.8.1 Causality

- The impulse response of an LTI system is zero for $n < 0$
- The impulse response of a causal LTI system is determined from the transfer function by using right sided inverse transforms
- The pole inside the unit circle in the z-plane contributes an exponentially decaying term
- The pole outside the unit circle in the z-plane contributes an exponentially increasing term

1.8.2 Stability

- The system is stable: if impulse response is absolutely summable and DTFT of impulse response exists
- The ROC must include the unit circle: the pole and unit circle together define the behavior of the system

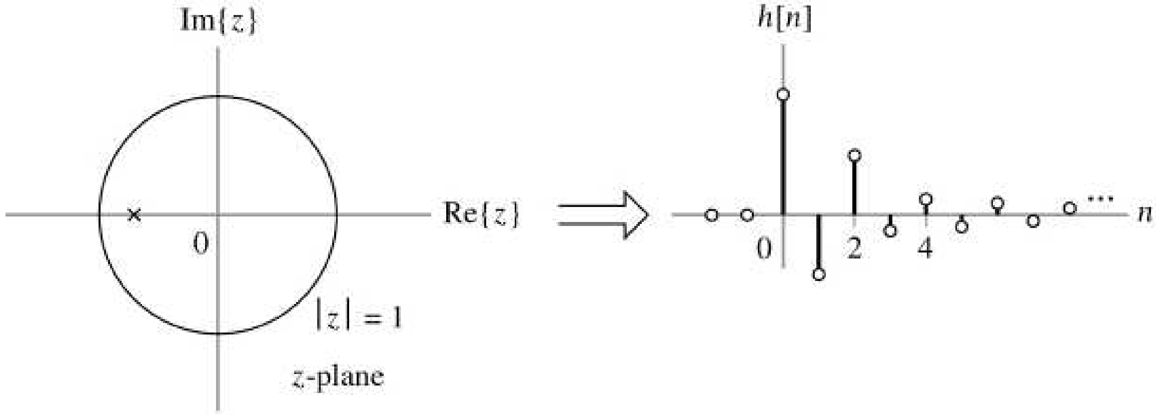


Figure 1.47: When the pole is inside the unit circle

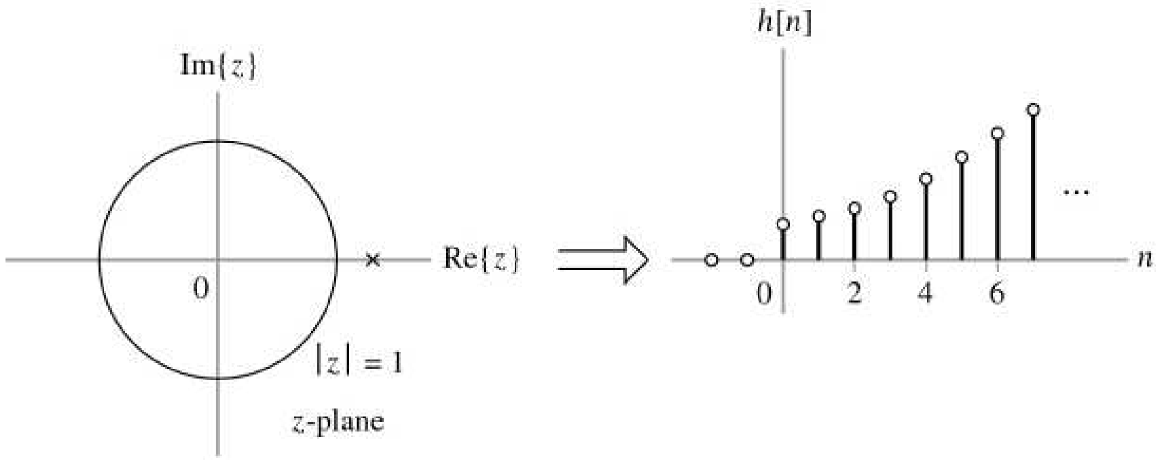


Figure 1.48: When the pole is outside the unit circle

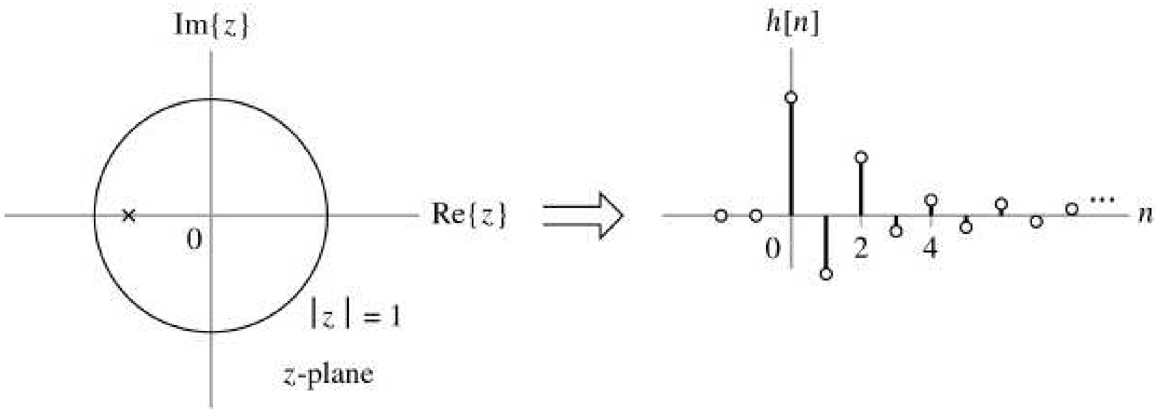


Figure 1.49: Stability: When the pole is inside the unit circle

- A stable impulse response can not contain any increasing exponential term
- The pole inside the unit circle in the z-plane contributes right-sided exponentially decaying term
- The pole outside the unit circle in the z-plane contributes left-sided exponentially decaying term

1.8.3 Causal and stable system

- Stable and causal LTI system: all the poles must be inside the unit circle
- A inside pole contributes right sided or causal exponentially decaying system
- A outside pole contributes either left sided decaying term which is not causal or right-sided exponentially increasing term which is not stable

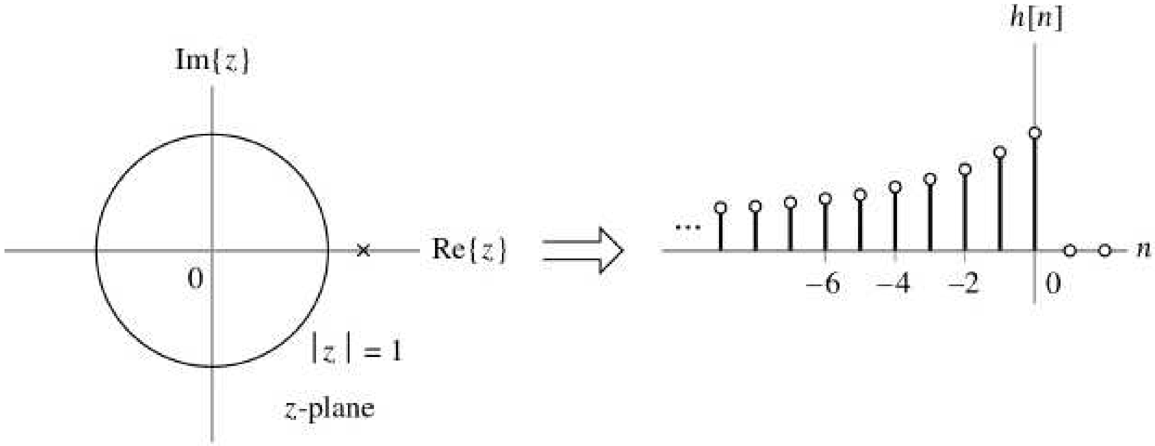


Figure 1.50: Stability: When the pole is outside the unit circle

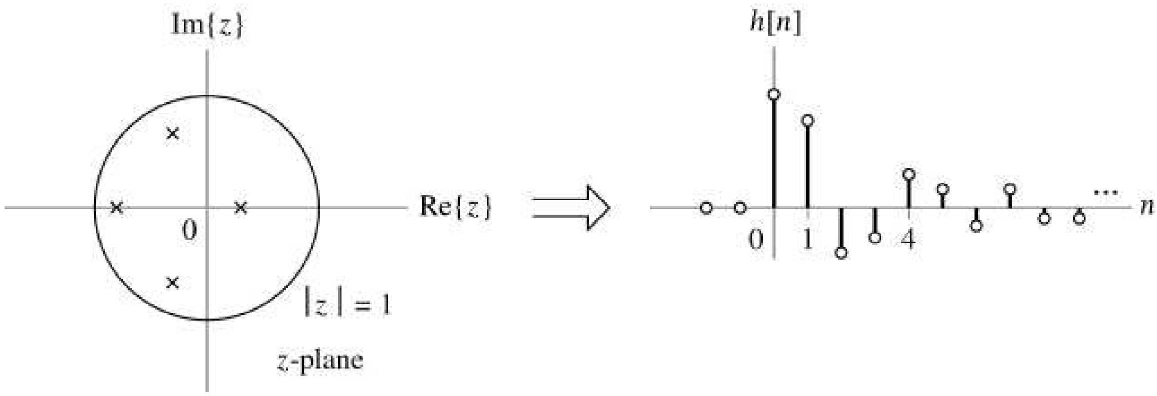


Figure 1.51: Location of poles for the causal and stable system

- Example of stable and causal system: all the poles are inside the unit circle

1.8.4 Examples

Ex1: Causal and Stable

- Find the impulse response when (a) system is stable (b) causal (c) can

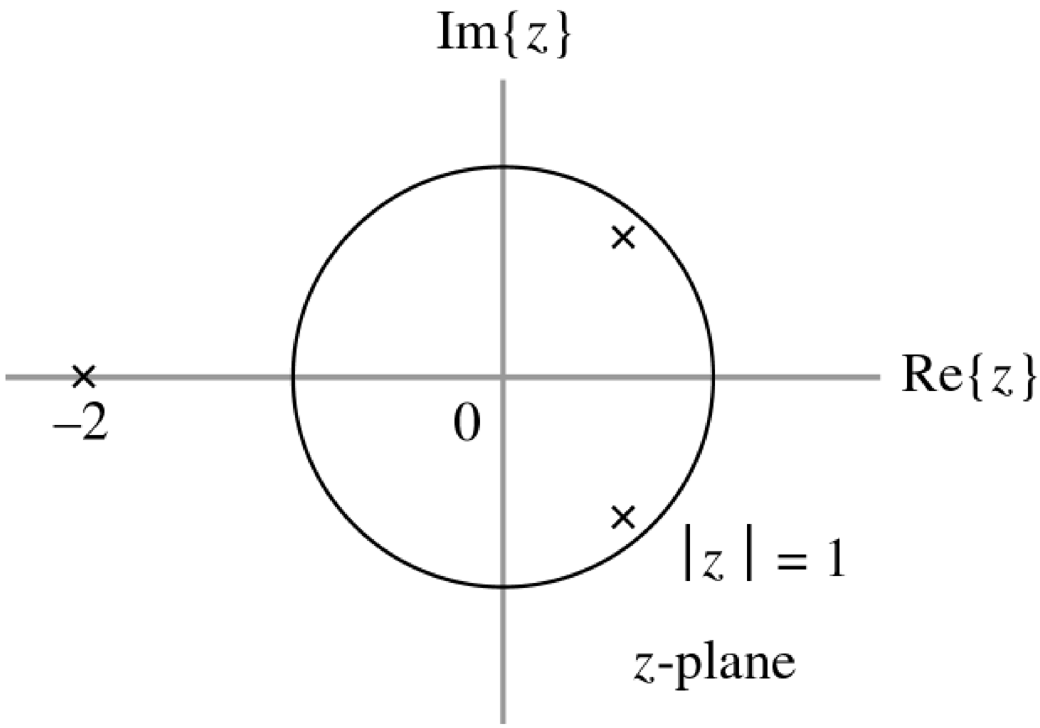


Figure 1.52: Location of poles in Example 1

this system be causal and stable?

$$H(z) = \frac{2}{1-0.9e^{j\frac{\pi}{4}}z^{-1}} + \frac{2}{1-0.9e^{-j\frac{\pi}{4}}z^{-1}} + \frac{3}{1+2z^{-1}}$$

- Solution: The system has poles at $z = e^{j\frac{\pi}{4}}$ and $z = e^{-j\frac{\pi}{4}}$ and $z = -2$

Ex1(a): Stable system

- The location of poles in the z-plane

Ex1(a): Stable system

- For stable system: the ROC must include the unit circle
- Two poles inside the unit circle contribute right sided terms

- The pole outside the unit circle gives left sided sequence

Ex1(a): Stable system

- Impulse response is

$$h[n] = 2(0.9e^{j\frac{\pi}{4}})^n u[n] + 2(0.9e^{-j\frac{\pi}{4}})^n u[n]$$

$$-3(-2)^n u[-n-1]$$

$$h[n] = 4(0.9)^n \cos(\frac{\pi}{4}n)u[n]$$

$$-3(-2)^n u[-n-1]$$

Ex1(b): Causal system

- For causal system: all the poles must contribute the right sided terms,

$$h[n] = 2(0.9e^{j\frac{\pi}{4}})^n u[n] + 2(0.9e^{-j\frac{\pi}{4}})^n u[n]$$

$$+3(-2)^n u[n]$$

$$h[n] = 4(0.9)^n \cos(\frac{\pi}{4}n)u[n] + 3(-2)^n u[n]$$

Ex1(c): Causal and stable system

- For causal and stable system: all the poles must be inside the unit circle
- We have one pole at $z = -2$, which is outside the unit circle, hence the system can not be both stable and causal system

Ex2: Recursive system

- Find the given recursive system is both stable and causal $y[n] - \rho y[n-1] = x[n]$ with $\rho = 1 + \frac{r}{100}$ and r is +ve
- $H(z) = \frac{1}{1-\rho z^{-1}}$, a pole is at $z = \rho$ and is greater than one, the pole is outside the unit circle and hence system can not be both causal and stable system

Ex3: Stable and causal system

- Find the impulse response of a stable and causal system described by a difference equation

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = -2x[n] + \frac{5}{4}x[n-1]$$

- Solution: find the z-transform from the difference equation and then find the impulse response

Ex3: Stable and causal system

- z-transform is

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$H(z) = \frac{-2 + \frac{5}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

- Find the poles of $H(z)$ and Write the denominator in product form

Ex3: Stable and causal system

- We get

$$H(z) = \frac{-2 + \frac{5}{4}z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

- Write $H(z)$ in terms of partial fraction expansion

$$H(z) = \frac{-3}{1+\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}}$$

Ex3: Stable and causal system

- Now write the impulse response for a causal and stable system

$$h[n] = -3(-\frac{1}{2})^n u[n] + (\frac{1}{4})^n u[n]$$

1.8.5 Inverse system

- Impulse response (h^{inv}) of an inverse system satisfies

$$h^{inv}[n] * h[n] = \delta[n]$$

  where $h[n]$ is the impulse response of a system to be inverted

- Take inverse z-transform on both sides gives

$$H^{inv}(z)H(z) = 1$$

$$H^{inv}(z) = \frac{1}{H(z)}$$

- The transfer function of an LTI inverse system is the inverse of the transfer function of the system that we desire to invert

- If we write the pole-zero form of $H(z)$ as

$$H(z) = \frac{\tilde{b}z^{-p}\prod_{k=1}^{M-p}(1-c_k z^{-1})}{z^{-l}\prod_{k=1}^{N-l}(1-d_k z^{-1})}$$

where $\tilde{b} = b_p / a_l$

- Then we can write H^{inv} as

$$H^{inv}(z) = \frac{z^{-l} \prod_{k=1}^{N-l} (1 - d_k z^{-1})}{\tilde{b} z^{-p} \prod_{k=1}^{M-p} (1 - c_k z^{-1})}$$

- The zeros of $H(z)$ are the poles of $H^{inv}(z)$
- The poles of $H(z)$ are the zeros of $H^{inv}(z)$
- System defined by a rational transfer function has an inverse system
- We need inverse systems which are both stable and causal to invert the distortions introduced by the system
- The inverse system $H^{inv}(z)$ is stable and causal if all poles are inside the unit circle
- Poles of $H^{inv}(z)$ are zeros of (z)
- Inverse system $H^{inv}(z)$: stable and causal inverse of an LTI system $H(z)$ exists *if and only if* all the zeros of $H(z)$ are inside the unit circle
- The system with all its poles and zeros inside the unit circle is called as *minimum-phase* system
- The magnitude response is uniquely determined by the phase response and vice-Vera
- For a *minimum-phase* system the magnitude response is uniquely determined by the phase response and vice-versa

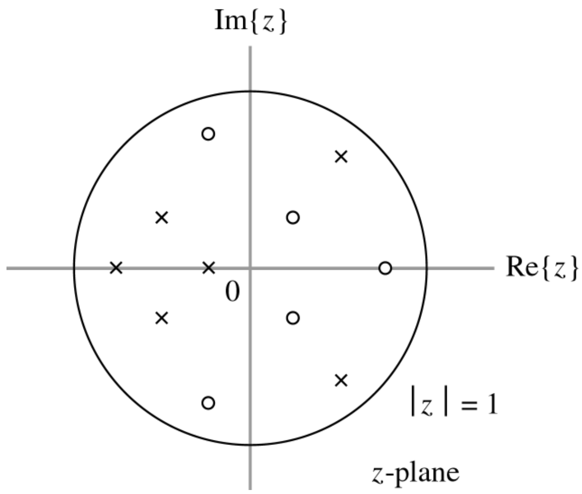


Figure 1.53: Location of poles in a *minimum-phase* system

Ex1: stable and causal system

- Find the transfer function of an inverse LTI system described by a difference equation

$$y[n]-y[n-1]+\frac{1}{4}y[n-2]=x[n]+\frac{1}{4}x[n-1]-\frac{1}{8}x[n-2]$$

  Is the system stable and causal?

- Solution: find the z-transform from the difference equation and then find the impulse response

- z-transform is

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$H(z)=\frac{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}{1-z^{-1}+\frac{1}{4}z^{-2}}$$

- Find the poles and zeros of $H(z)$ and write in the product form

- We can write $H(z)$ as

$$H(z) = \frac{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})^2}$$

- We can write $H^{inv}(z)$ as

$$H^{inv}(z) = \frac{1}{H(z)} = \frac{(1-\frac{1}{2}z^{-1})^2}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})}$$

- The poles of inverse system are at $z=\frac{1}{4}$ and $z=-\frac{1}{2}$
- Two poles are inside the unit circle, hence the system can be both stable and causal
- The zero is also inside the unit circle and the system is *minimum-phase* system

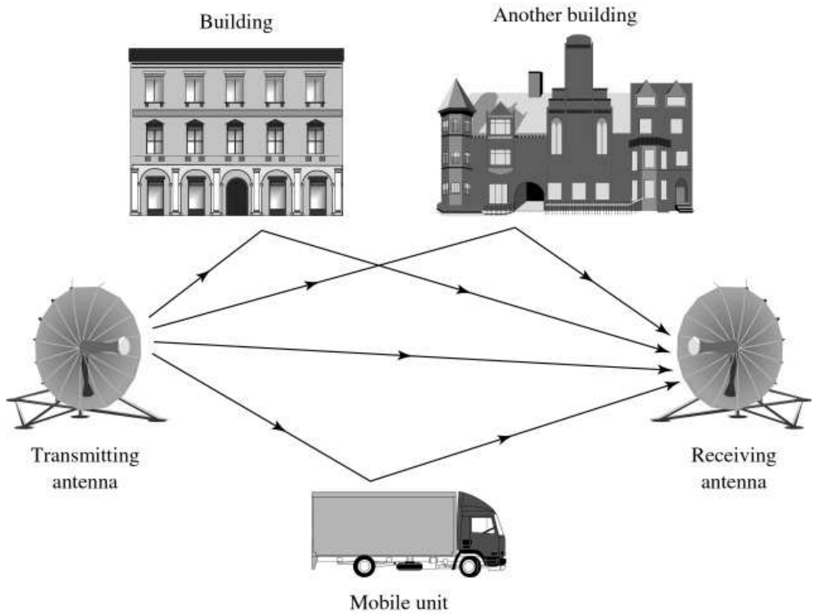
Ex2: inverse system

- Find the transfer function and difference equation description of the inverse system of discrete LTI system, which describes multipath communication channel (two path communication channel)

$$y[n] = x[n] + ax[n-1]$$

- The system is
- z-transform is

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$



$$H(z) = 1 + az^{-1}$$

- The inverse system is

$$H^{inv}(z) = \frac{1}{H(z)} = \frac{1}{1 + az^{-1}}$$

- The corresponding difference equation description is

$$y[n] + ay[n-1] = x[n]$$

- The inverse system is both stable and causal when $|a| < 1$

Ex3: inverse system

- Find the transfer function of the inverse system and is the inverse system stable and causal?

$$h[n] = 2\delta[n] + \frac{5}{2}(\frac{1}{2})^n u[n] - \frac{7}{2}(-\frac{1}{4})^n u[n]$$

$$H(z) = 2 + \frac{5}{2}\frac{1}{(1-\frac{1}{2}z^{-1})} - \frac{7}{2}\frac{1}{(1+\frac{1}{4}z^{-1})}$$

- Write $H(z)$ in product form

$$H(z) = \frac{2(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1}) + \frac{5}{2}(1+\frac{1}{4}z^{-1}) + \frac{7}{2}(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$$

- After simplification we get

$$H(z) = \frac{(1-\frac{1}{8}z^{-1})(1+2z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$$

- We know $H^{inv}(z) = \frac{1}{H(z)}$, so we can write

$$H^{inv}(z) = \frac{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}{(1-\frac{1}{8}z^{-1})(1+2z^{-1})}$$

- Poles are at $z = \frac{1}{8}$ and $z = -2$, so the system can not be both stable and causal

1.8.6 Unsolved examples from [2]

Unsolved ex:7.34(a)

- Is the system (i)causal and stable (ii)minimum phase

$$H(z) = \frac{2z+3}{z^2+z-\frac{5}{16}}$$

$$H(z) = \frac{2z+3}{(z+\frac{5}{4})(z-\frac{1}{4})}$$

- Write the poles and zeros
- Poles are at $z = \frac{1}{8}$ and $z = -2$, so the system can not be both stable and causal
- Poles are at $z = \frac{1}{4}$ and $z = -\frac{5}{4}$, and a zero is at $z = -\frac{3}{2}$
- (i) All poles are not inside $|z| = 1$, hence the system is not causal and stable.
- (ii) All poles and zeros are not inside $|z| = 1$, the system is not minimum phase.

Unsolved ex:7.34(b)

- Is the system (i)causal and stable (ii)minimum phase

$$y[n] - y[n-1] - \frac{1}{4}y[n-2] = 3x[n] - 2x[n-1]$$

- Write the poles and zeros
- Poles are at $z = \frac{1 \pm \sqrt{(2)}}{2}$ and zeros are at $z = 0$ and $z = \frac{2}{3}$
- (i) All poles are not inside $|z| = 1$, hence the system is not causal and stable.
- (ii) All poles and zeros are not inside $|z| = 1$, the system is not minimum phase.
- (i) All poles are not inside $|z| = 1$, hence the system is not causal and stable.

- (ii) All poles and zeros are not inside $|z| = 1$, the system is not minimum phase.

Unsolved ex:7.35(a)

- Find the transfer function of an inverse system and determine whether it can be both causal and stable

$$H(z) = \frac{1 - 8z^{-1} + 16z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}$$

$$H(z) = \frac{(z-4)^2}{(z-\frac{1}{2})^2}$$

- Write the inverse system

$$H^{inv}(z) = \frac{(z-\frac{1}{2})^2}{(z-4)^2}$$

- Double poles at $z = 4$, hence the poles are outside the unit circle and inverse system is not stable and causal

Unsolved ex:7.35(b)

- Find the transfer function of an inverse system and determine whether it can be both causal and stable

$$H(z) = \frac{z^2 - \frac{81}{100}}{z^2 - 1}$$

$$H^{inv}(z) = \frac{z^2 - 1}{z^2 - \frac{81}{100}}$$

- Double poles at $z = \frac{9}{10}$, hence the poles are inside the unit circle and inverse system is stable and causal

Conclusions

- Causality of an LTI system and the inverse z-transform
- Stability of an LTI system and the inverse z-transform
- Inverse systems
- Stability and causality of an inverse system

1.9 Class 9: Pole zero representation and frequency response

Outline of today's class

- Pole zero representation of $H(z)$
- Determining the frequency response from poles and zeros

1.9.1 Frequency response

- The frequency response is obtained from the z-transform by substituting $z = e^{j\Omega}$ in $H(z)$
- The frequency response is transfer function evaluated on the unit circle in the z-plane
- We assume existence of DTFT ie. ROC includes the unit circle
- Substituting $z = e^{j\Omega}$ in the pole-zero representation of $H(z)$, and is given by

$$H(e^{j\Omega}) = \frac{\tilde{b}e^{-jp\Omega}\prod_{k=1}^{M-p}(1 - c_k e^{-j\Omega})}{e^{-jl\Omega}\prod_{k=1}^{N-l}(1 - d_k e^{-j\Omega})}$$

- Rewrite $H(e^{j\Omega})$ in terms of +ve powers of $e^{j\Omega}$, this is done by multiplying $e^{jN\Omega}$ to both numerator and denominator
- We get

$$H(e^{j\Omega}) = \frac{\tilde{b}e^{j(N-M)\Omega}\prod_{k=1}^{M-p}(e^{j\Omega} - c_k)}{\prod_{k=1}^{N-l}(e^{j\Omega} - d_k)}$$

- Examine the magnitude and phase response of $H(e^{j\Omega})$
- Evaluate the magnitude of $H(e^{j\Omega})$ at some fixed value of Ω_o
- Magnitude of $H(e^{j\Omega})$ at Ω_o is given by

$$H(e^{j\Omega_o}) = \frac{|\tilde{b}|\prod_{k=1}^{M-p}|e^{j\Omega} - c_k|}{\prod_{k=1}^{N-l}|e^{j\Omega} - d_k|}$$

- This consists of a ratio of products of the terms $|e^{j\Omega} - g|$, where g is either a pole or zero. We have terms with zeros in the numerator and poles in the denominator
- Use vectors to represent the complex numbers in the z-plane
- $e^{j\Omega_o}$ is a vector from origin to the point $e^{j\Omega_o}$
- g is a vector from origin to the point g
- $e^{j\Omega_o} - g$ is represented as a vector from the point g to the point $e^{j\Omega_o}$
- The vectors are
- The length of vector is $|e^{j\Omega_o} - g|$
- Asses the contribution of each pole and zero to the overall frequency response by examining $|e^{j\Omega_o} - g|$ as Ω_o changes
- Find $|e^{j\Omega_o} - g|$ for different values of Ω_o
- The different vectors are
- Figure 1.55(a) shows the vector $e^{j\Omega_o} - g$ for different values of Ω

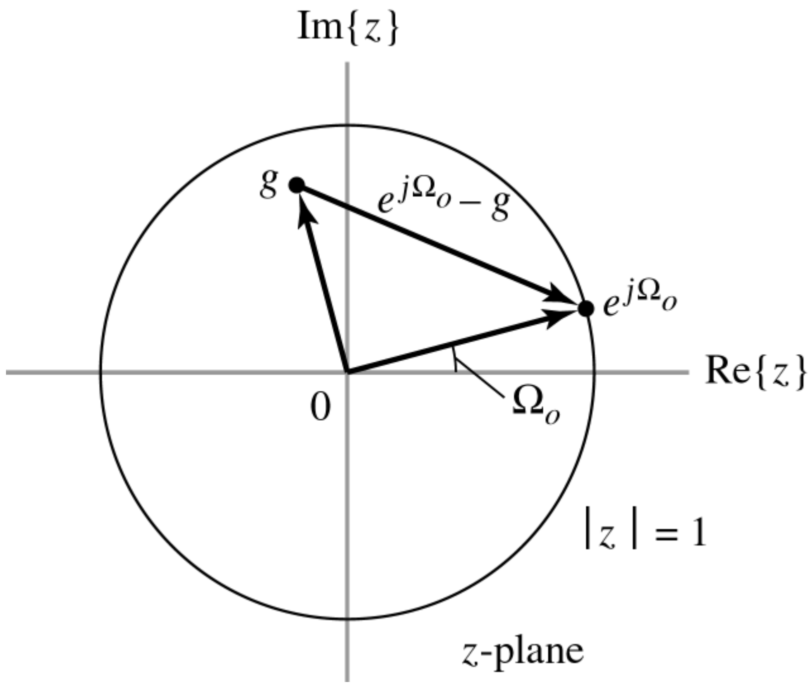


Figure 1.54: Vector g and $e_o^{j\Omega}$ in the z-plane

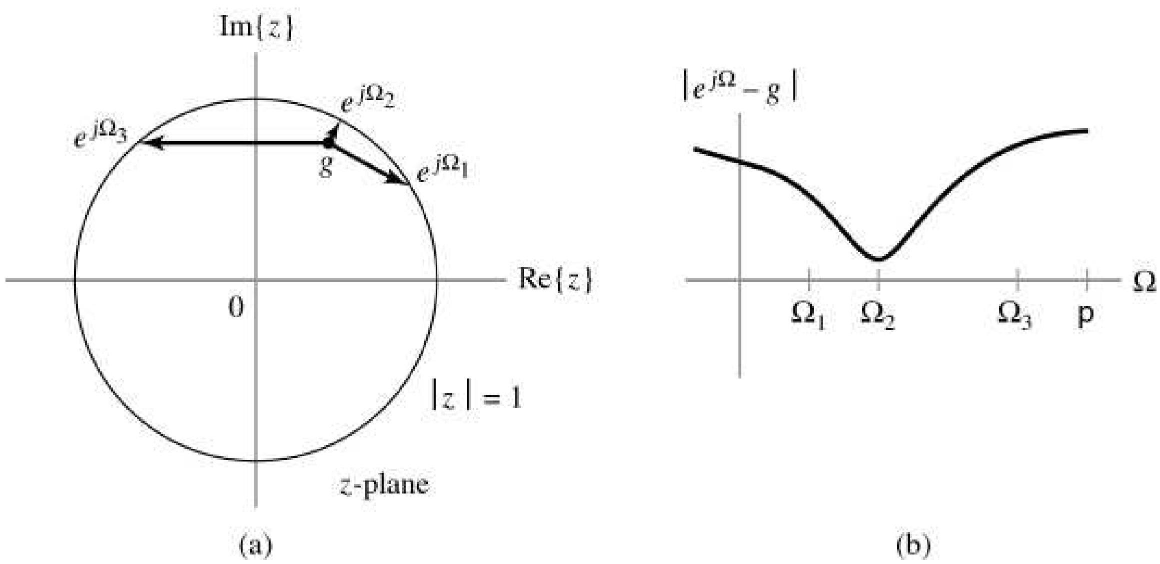


Figure 1.55: $|e^{j\Omega_o} - g|$ for different values of Ω

- Figure 1.55(b) shows $|e^{j\Omega_o} - g|$ as a continuous function of frequency
- If $\Omega = arg\{g\}$, then $|e^{j\Omega_o} - g|$ takes a minimum value of $1 - |g|$ when g is inside the unit circle
- If $\Omega = arg\{g\}$, then $|e^{j\Omega_o} - g|$ takes a maximum value $|g| - 1$ when g is outside the unit circle
- If $\Omega = arg\{g\}$, and g is close to the unit circle $(g \approx 1)$, then $|e^{j\Omega_o} - g|$ becomes very small
- $|e^{j\Omega_o} - g|$ contributes to the numerator of $|H(e^{j\Omega_o})|$
- $|H(e^{j\Omega_o})|$ tends to take minimum value at frequencies near $arg\{g\}$
- How for $|H(e^{j\Omega_o})|$ decreases depends on how close it is to the unit circle
- If g is on the unit circle then $|H(e^{j\Omega_o})| = 0$ at g
- $|e^{j\Omega_o} - g|$ contributes to the denominator of $|H(e^{j\Omega_o})|$
- When $|e^{j\Omega_o} - g|$ decreases then $|H(e^{j\Omega_o})|$ increases depending on how far the pole is from the unit circle
- Close to the unit circle g causes large variation in $|H(e^{j\Omega_o})|$ at the frequency of g
- The zero tends to pull the frequency and pole tends to push the frequency response
- The zero tends to pull the frequency magnitude down at the frequency corresponding to zero

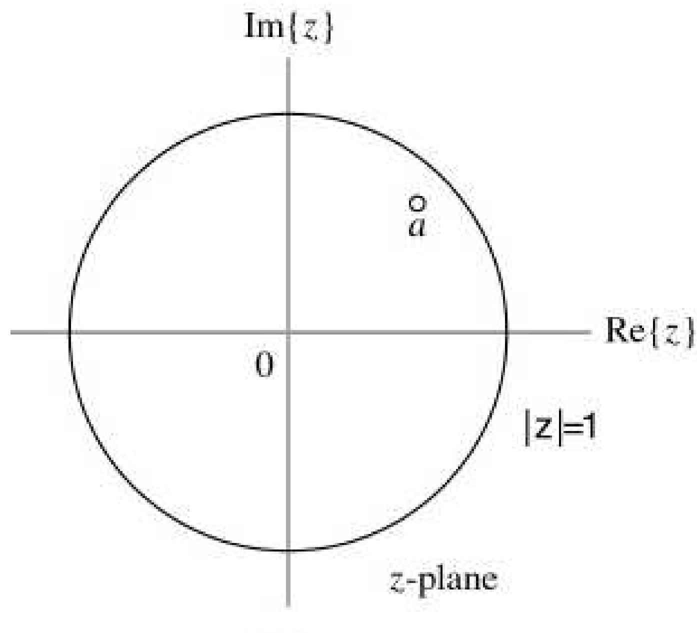


Figure 1.56: Zero location of the system

- The pole tends to push the frequency magnitude up at the frequency corresponding to pole

Ex1. Two path channel

- The two path communication channel is given by

$$H(z) = 1 + az^{-1}$$

- Sketch the magnitude response of the system and its inverse system for $a = 0.5e^{\frac{j\pi}{4}}$, $a = 0.8e^{\frac{j\pi}{4}}$ and $a = 0.95e^{\frac{j\pi}{4}}$
- The zero location of the system
- The two path communication channel has a zero at $z = a$

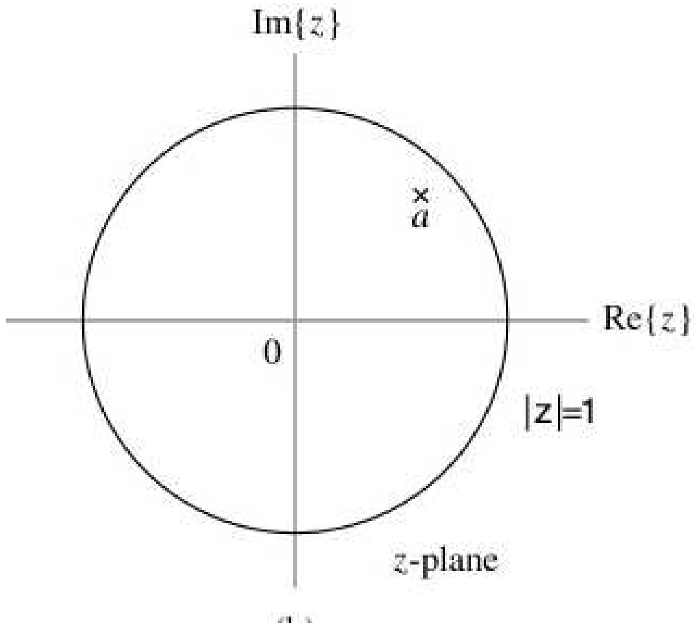


Figure 1.57: Pole location of the inverse system

- The minimum of $H(e^{j\Omega})$, for the system occurs at the frequency corresponding to the angle of zero of $H(z)$ at $\Omega = \pi/4$
- The minimum of $|H(e^{j\Omega})|$ is $1 - |a|$
- As a approaches unity, the channel magnitude response at $\Omega = \pi/4$ approaches to zero and the two path channel suppresses any components of the input having frequency at $\Omega = \pi/4$
- The pole location of the inverse system
- The inverse system has a pole at $z = a$
- The maximum of $H^{inv}(e^{j\Omega})$, for the inverse system occurs at the frequency corresponding to the angle of zero of $H(z)$ at $\Omega = \pi/4$
- The maximum of $|H^{inv}(e^{j\Omega})|$ occurs at $\Omega = \pi/4$, and is $\frac{1}{(1-|a|)}$

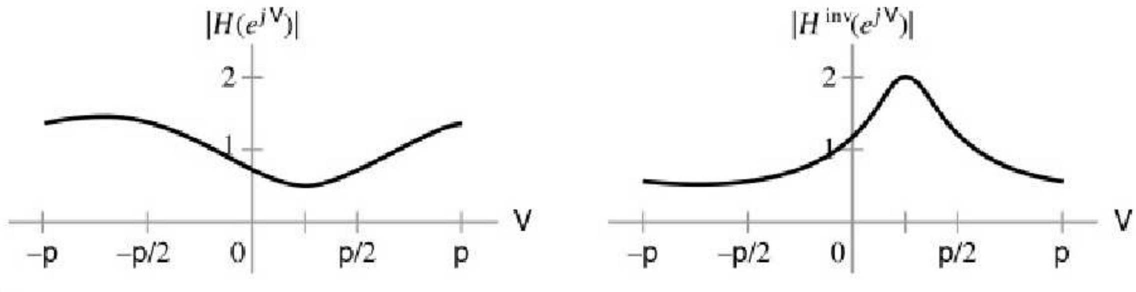


Figure 1.58: Magnitude response of the system and inverse system when $a = 0.5e^{\frac{j\pi}{4}}$

- As a approaches unity, the channel magnitude response of the inverse system at $\Omega = \pi/4$ approaches to infinity
- If the two path channel eliminates the components of the input having frequency at $\Omega = \pi/4$, the inverse system can not restore this component to original value
- Large value of gain are undesirable since it also enhances the noise. $H^{inv}9z)$ is highly sensitive to small changes in a as $a \rightarrow 1$
- The magnitude response for the system and inverse system when $a = 0.5e^{\frac{j\pi}{4}}$
- The magnitude response for the system and inverse system when $a = 0.8e^{\frac{j\pi}{4}}$
- The magnitude response for the system and inverse system when $a = 0.95e^{\frac{j\pi}{4}}$

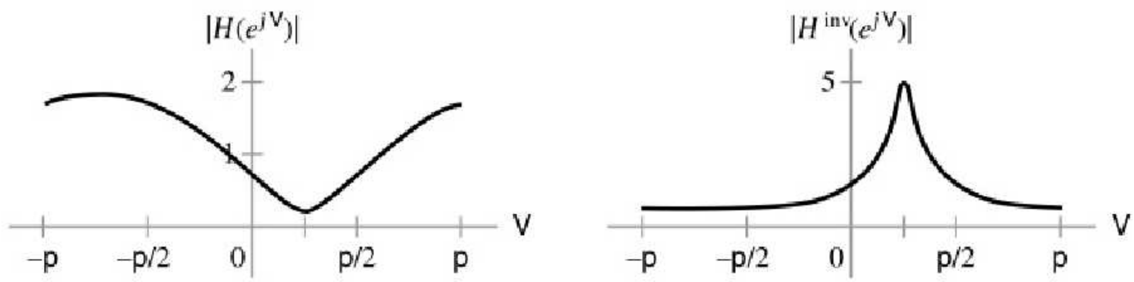


Figure 1.59: Magnitude response of the system and inverse system when $a = 0.8e^{\frac{j\pi}{4}}$

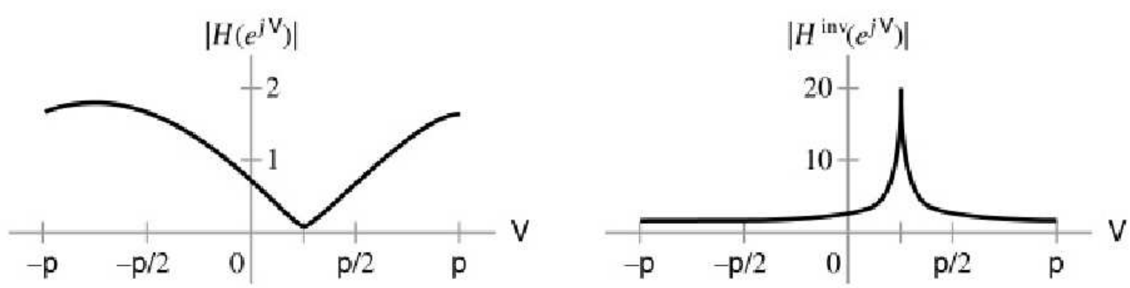


Figure 1.60: Magnitude response of the system and inverse system when $a = 0.95e^{\frac{j\pi}{4}}$

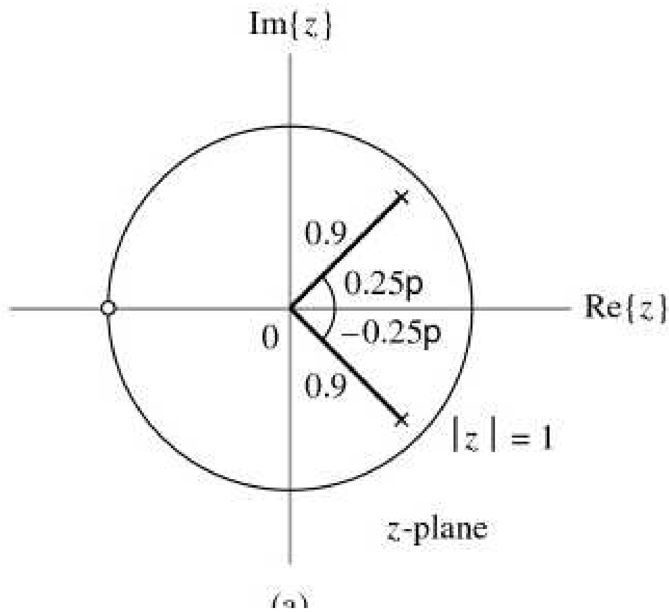


Figure 1.61: Pole zero location for Ex2

Ex2. Magnitude response

- Sketch the magnitude response of an LTI system with transfer function

$$H(z) = \frac{1+z^{-1}}{(1-0.9e^{\frac{j\pi}{4}}z^{-1})(1-0.9e^{\frac{-j\pi}{4}}z^{-1})}$$

- The system has zero at $z=-1$ and poles at $z=0.9e^{\frac{j\pi}{4}}$ and $z=0.9e^{\frac{-j\pi}{4}}$
- The magnitude response due to zero, the system has zero magnitude response at $\Omega=\pi$
- The magnitude response due to pole $z=0.9e^{\frac{j\pi}{4}}$, large magnitude response at $\Omega=\pm\pi/4$, pole is close to $|z|=1$

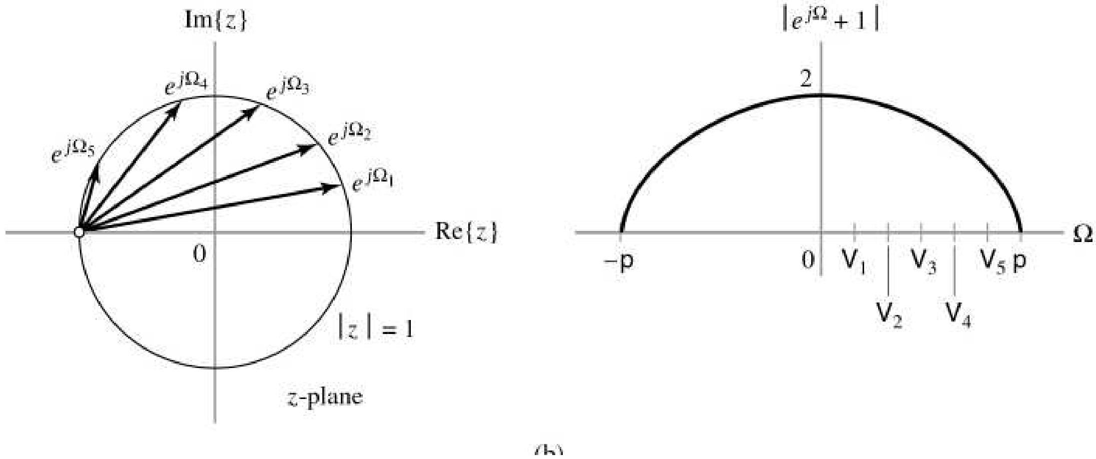


Figure 1.62: Magnitude response due to zero

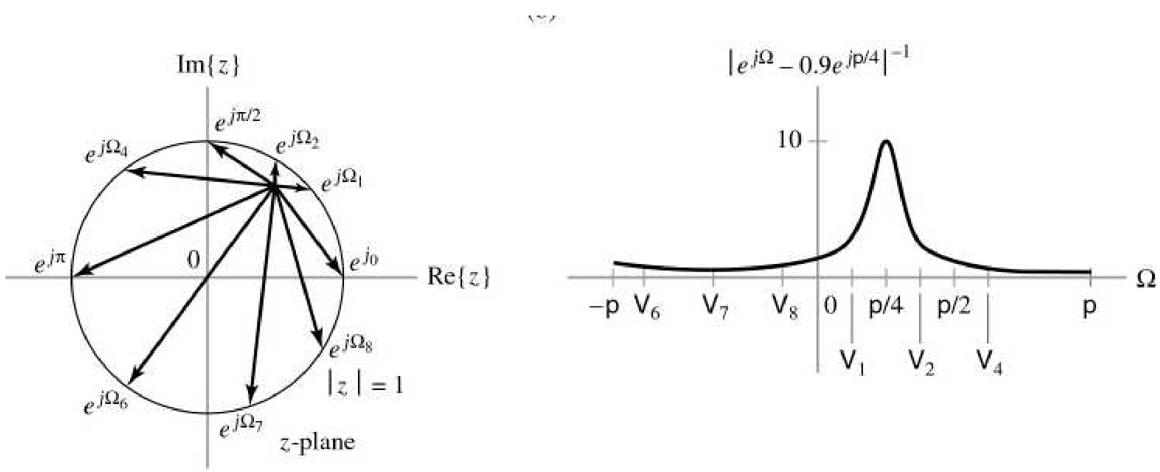


Figure 1.63: Magnitude response due to pole $z = 0.9e^{\frac{j\pi}{4}}$

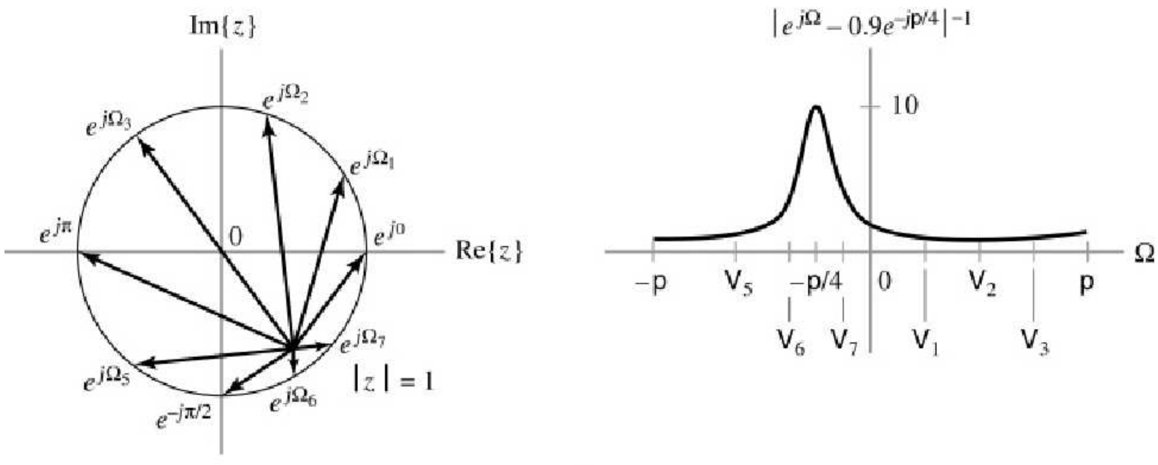


Figure 1.64: Magnitude response due to pole $z = 0.9e^{-\frac{j\pi}{4}}$

- The magnitude response due to pole $z = 0.9e^{-\frac{j\pi}{4}}$, large magnitude response at $\Omega = \pm\pi/4$, pole is close to $|z| = 1$
- The magnitude response due both poles and zero (product of all responses)

Ex3. Magnitude response

- Sketch the magnitude response of the LTI system with transfer function

$$H(z) = \frac{z-1}{z+0.9}$$

- The phase of $H(e^{j\Omega})$ is evaluated in terms of phase associated with each pole and zero
- We write the phase as

$$arg\{H(e^{j\Omega})\} = arg\{\tilde{b}\} + (N-M)\Omega$$

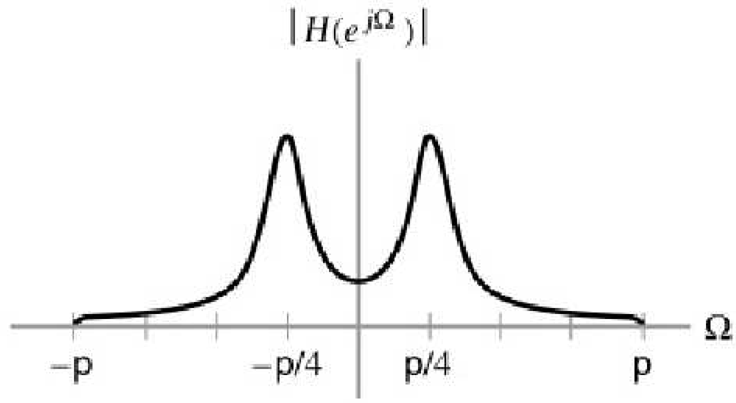


Figure 1.65: Overall magnitude response for Ex2

$$+\sum_{k=1}^{M-p} arg\{e^{j\Omega} - c_k\} - \sum_{k=1}^{N-l} arg\{e^{j\Omega} - d_k\}$$

- The phase of $H(e^{j\Omega})$=sum of phases due to zeros - sum of phases due to poles
- The first tern is independent of frequency
- The phase associated with each zero and pole is evaluated from $arg\{e^{j\Omega} - g\}$
- This is the angle associated with a vector pointing from g to $e^{j\Omega}$
- The angle is measured with respect to horizontal line passing through g
- The contribution of pole or zero to the overall response is determined by the angle of $e^{j\Omega} - g$ vector as the frequency changes

Frequency response

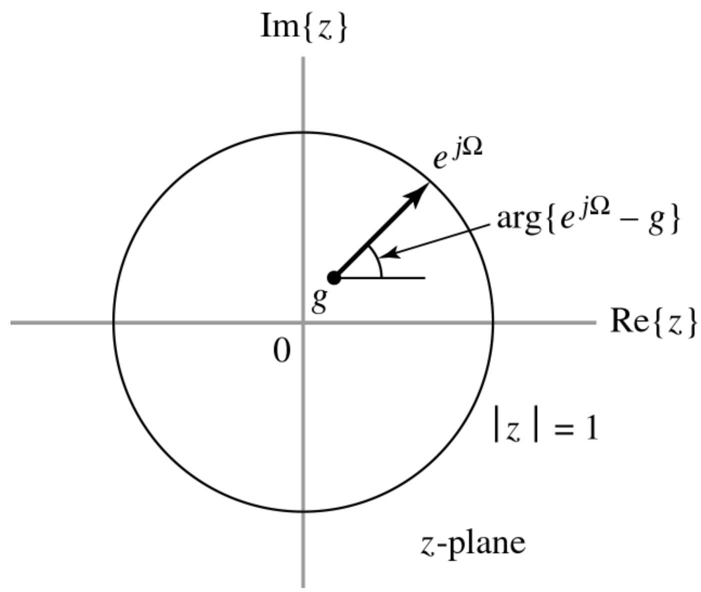


Figure 1.66: Phase associated with the pole or zero

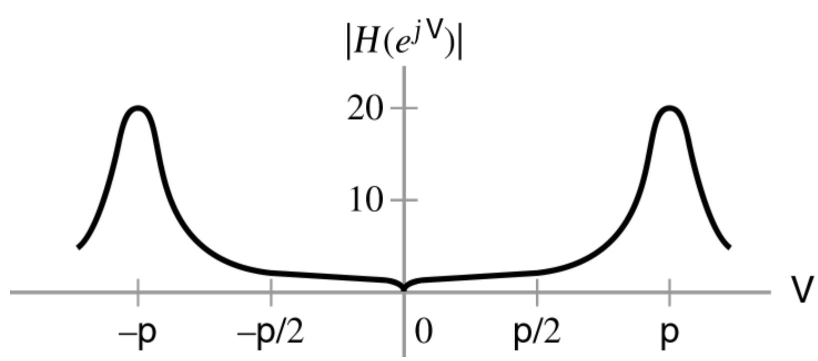


Figure 1.67: Overall magnitude response for Ex3

- Exact evaluation of frequency response is best performed numerically
- We can approximately estimate from the location of poles and zeros and get an insight into the nature of frequency response
- Asymptotic approximations like Bode-plots are not used for discrete time systems, since the frequency range is limited to $-\pi < \Omega < \pi$

1.9.2 Unsolved examples from [2]

Unsolved ex. 7.37(a)

- Sketch the magnitude response of the systems having the following transfer functions:

$$H(z) = \frac{z^{-2}}{1 + \frac{49}{64}z^{-2}}$$

- Solution: Write $H(z)$ in the product form

$$H(z) = \frac{1}{(z + j\frac{7}{8})(z - j\frac{7}{8})}$$

$$H(e^{j\Omega}) = \frac{1}{(e^{j\Omega} + j\frac{7}{8})(e^{j\Omega} - j\frac{7}{8})}$$

- The locations of poles in the z-plane
- The magnitude response is

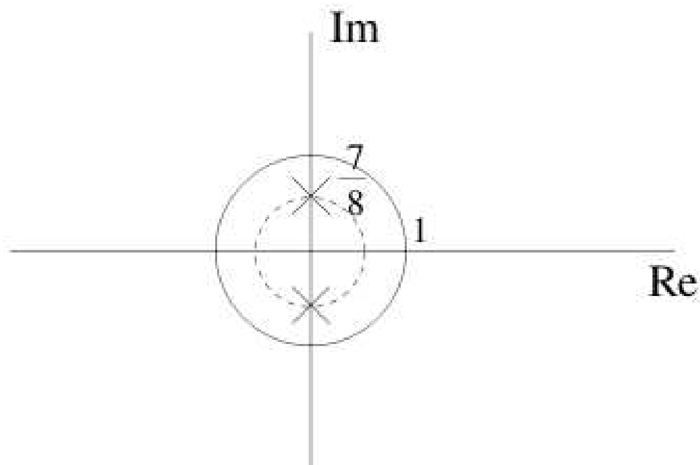


Figure 1.68: Pole zero locations in the z-plane

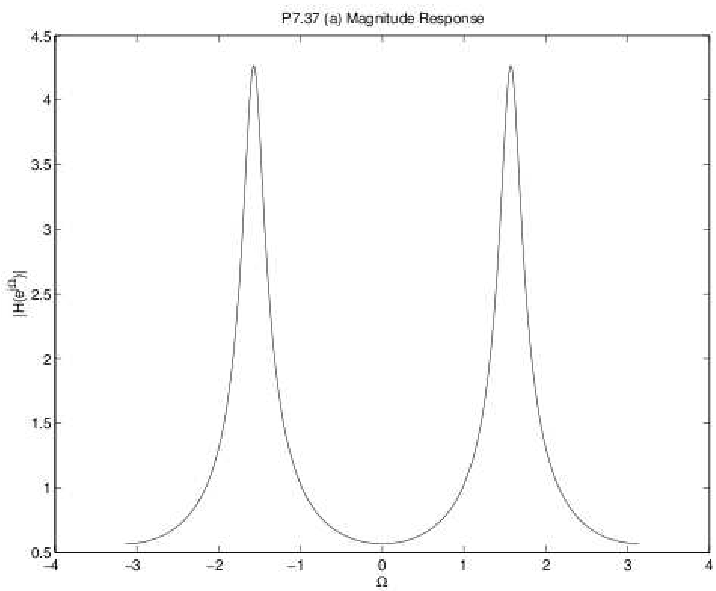


Figure 1.69: Magnitude response

Unsolved ex. 7.37(b)

- Sketch the magnitude response of the systems having the following transfer functions:
$$H(z) = \frac{1+z^{-1}+z^{-2}}{3}$$
- Solution: Write $H(z)$ in the product form
- We get
$$H(z) = \frac{z^2+z+1}{3z^2}$$
$$H((e^{j\Omega}) = \frac{e^{j2\Omega}+e^{j\Omega}+1}{3e^{j2\Omega}}$$
- Poles at $z = 0$ (double), and zeros at $z = e^{\pm\frac{j2\pi}{3}}$
- The locations of poles in the z-plane
- The magnitude response is

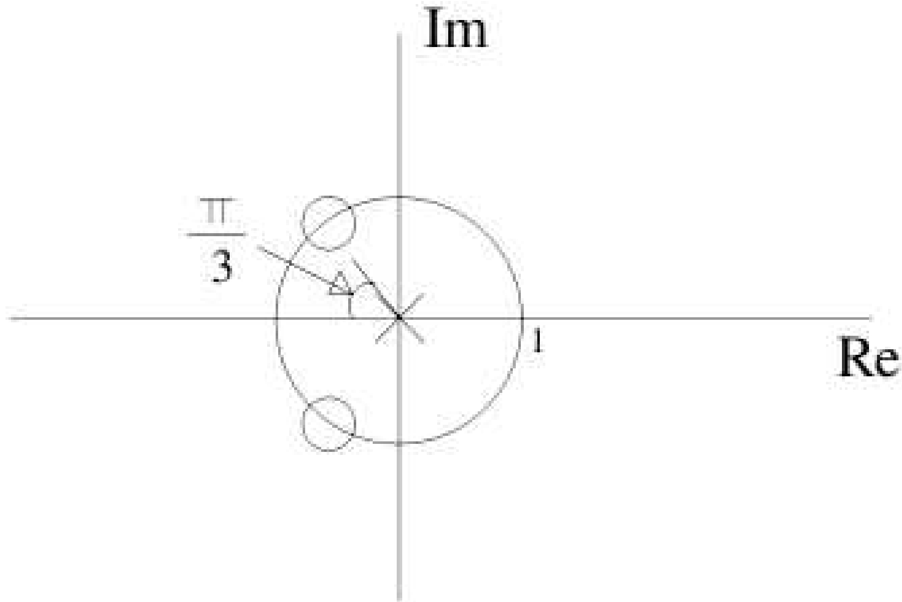


Figure 1.70: Pole zero locations in the *z*-plane

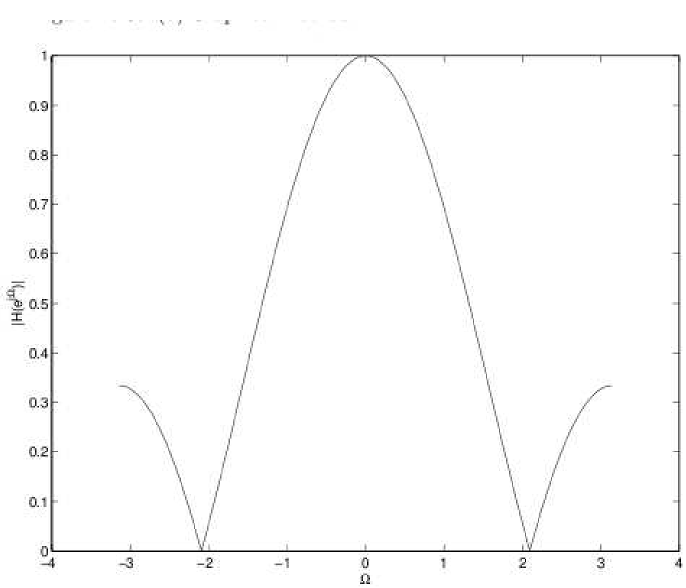


Figure 1.71: Magnitude response

Unsolved ex. 7.37(c)

- Sketch the magnitude response of the systems having the following transfer functions:

$$H(z) = \frac{1+z^{-1}}{1+\frac{18}{10}\cos(\frac{\pi}{4})z^{-1}+\frac{81}{100}z^{-2}}$$

- Solution: Write $H(z)$ in the product form and we get

$$H(z) = \frac{1+z^{-1}}{(1-\frac{9}{10}e^{\frac{j3\pi}{4}}z^{-1})(1-\frac{9}{10}e^{\frac{-j3\pi}{4}}z^{-1})}$$

$$H(e^{j\Omega}) = \frac{e^{j2\Omega}+e^{j\Omega}}{e^{j\Omega}+(18/10)\cos(\frac{\pi}{4})e^{j\Omega}+81/100}$$

- Zeros at $z=-1$ (double), and poles at $z=e^{\pm\frac{j3\pi}{4}}$
- The locations of poles in the z-plane
- The magnitude response is

Conclusions

- Pole zero representation of $H(z)$
- Determining the frequency response from poles and zeros

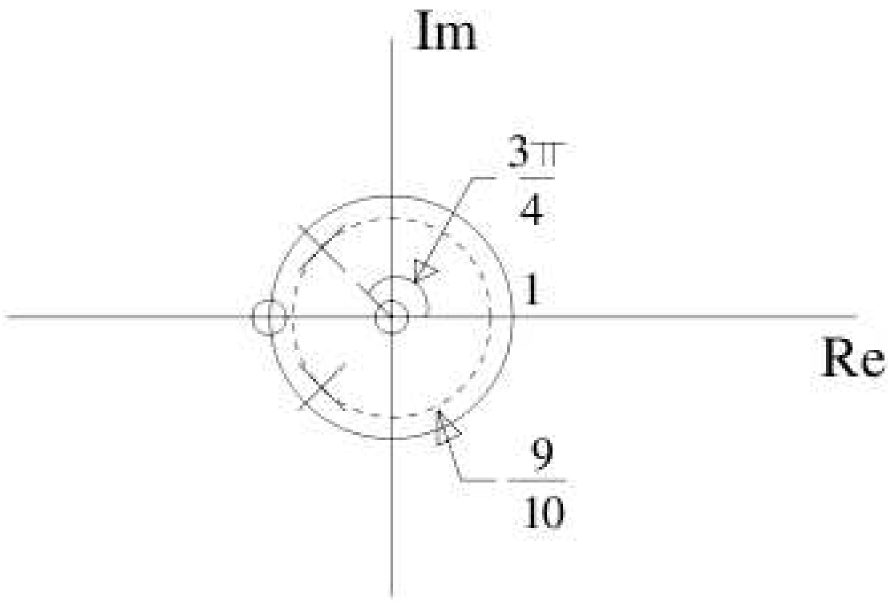


Figure 1.72: Pole zero locations in the z-plane

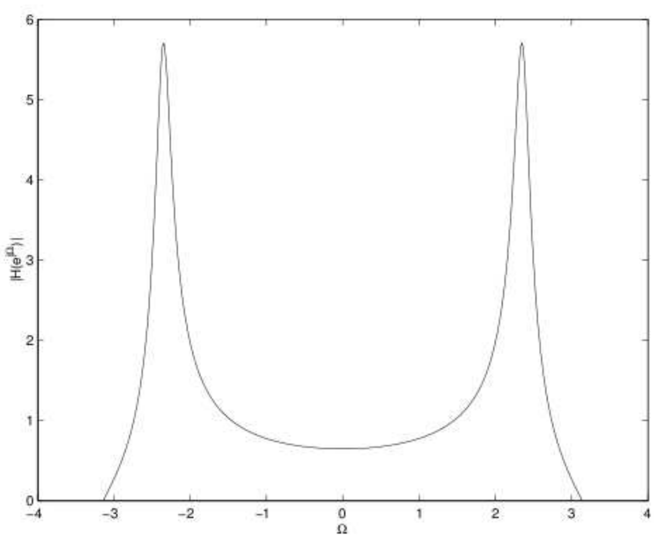


Figure 1.73: Magnitude response

1.10 Class 10: Unilateral z-transform

Outline of today's class

- Unilateral z-transform
- Properties of unilateral z-transform
- Solving difference equation with initial condition
- Implementations of continuous time systems

1.10.1 Unilateral z-transform

- Useful in case of causal signals and LTI systems
- The choice of time origin is arbitrary, so we may choose $n = 0$ as the time at which the input is applied and then study the response for times $n \geq 0$

Advantages

- We do not need to use ROCs
- It allows the study of LTI systems described by the difference equation with initial conditions

Unilateral z-transform

- The unilateral z-transform of a signal $x[n]$ is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

which depends only on $x[n]$ for $n \geq 0$

- The unilateral and bilateral z-transforms are equivalent for causal signals

$$\alpha^n u[n] \stackrel{z_u}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}$$

$$a^n \cos(\Omega_o n)u[n] \stackrel{z_u}{\longleftrightarrow} \frac{1-a\cos(\Omega_o)z^{-1}}{1-2a\cos(\Omega_o)z^{-1}+a^2z^{-2}}$$

1.10.2 Properties

- The same properties are satisfied by both unilateral and bilateral z-transforms with one exception: the time shift property
- The time shift property for unilateral z-transform: Let $w[n] = x[n-1]$
- The unilateral z-transform of $w[n]$ is

$$W(z) = \sum_{n=0}^{\infty} w[n]z^{-n} = \sum_{n=0}^{\infty} x[n-1]z^{-n}$$

$$W(z) = x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n}$$

$$W(z) = x[-1] + \sum_{m=0}^{\infty} x[m]z^{-(m+1)}$$

- The unilateral z-transform of $w[n]$ is

$$W(z) = x[-1] + z^{-1}\sum_{m=0}^{\infty} x[m]z^{-m}$$

$$W(z) = x[-1] + z^{-1}X(z)$$

- A one-unit time shift results in multiplication by z^{-1} and addition of the constant $x[-1]$
- In a similar way, the time-shift property for delays greater than unity is

$$x[n-k] \stackrel{z_u}{\longleftrightarrow} x[-k] + x[-k+1]z^{-1} + \ldots + x[-1]z^{-k+1} + z^{-k}X(z) \quad \text{for} \quad k > 0$$

- In the case of time advance, the time-shift property changes to

$$x[n+k] \stackrel{z_u}{\longleftrightarrow} -x[0]z^{k} - x[-1]z^{k-1} + \ldots - x[k-1]z + z^{k}X(z) \quad \text{for} \quad k > 0$$

1.10.3 Solving difference equation with initial conditions

- Consider the difference equation description of an LTI system

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- We may write the z-transform as

$$A(z)Y(z) + C(z) = B(z)X(z)$$

where

$$A(z) = \sum_{k=0}^{N} a_k z^{-k} \quad \text{and} \quad B(z) = \sum_{k=0}^{M} b_k z^{-k}$$

- We get

$$C(z) = \sum_{m=0}^{N-1} \sum_{k=m+1}^{N} a_k y[-k+m] z^{-m}$$

- We have assumed that $x[n]$ is causal and

$$x[n-k] \xleftrightarrow{z_u} z^{-k} X(z)$$

- The term $C(z)$ depends on the N initial conditions $y[-1], y[-2], \ldots, y[-N]$ and the a_k
- $C(z)$ is zero if all the initial conditions are zero
- Solving for $Y(z)$, gives

$$Y(z) = \frac{B(z)}{A(z)} X(z) - \frac{C(z)}{A(z)}$$

- The output is the sum of the forced response due to the input and the natural response induced by the initial conditions
- The forced response due to the input

$$\frac{B(z)}{A(z)} X(z)$$

- The natural response induced by the initial conditions

$$\frac{C(z)}{A(z)}$$

- $C(z)$ is the polynomial, the poles of the natural response are the roots of $A(z)$, which are also the poles of the transfer function

- The form of natural response depends only on the poles of the system, which are the roots of the characteristic equation

First order recursive system

- Consider the first order system described by a difference equation

$$y[n] - \rho y[n-1] = x[n]$$

  where $\rho = 1 + r/100$, and r is the interest rate per period in percent and $y[n]$ is the balance after the deposit or withdrawal of $x[n]$

-

- Assume bank account has an initial balance of \$10,000/- and earns 6% interest compounded monthly. Starting in the first month of the second year, the owner withdraws \$100 per month from the account at the beginning of each month. Determine the balance at the start of each month.

- Solution: Take unilateral z-transform and use time-shift property we get

$$Y(z) - \rho(y[-1] + z^{-1}Y(z)) = X(z)$$

- Rearrange the terms to find $Y(z)$, we get

$$(1 - \rho z^{-1})Y(z) = X(z) + \rho y[-1]$$

$$Y(z) = \frac{X(z)}{1 - \rho z^{-1}} + \frac{\rho y[-1]}{1 - \rho z^{-1}}$$

-

- $Y(z)$ consists of two terms
  - one that depends on the input: the forced response of the system
  - another that depends on the initial conditions: the natural response of the system

- The initial balance of \$10,000 at the start of the first month is the initial condition $y[-1]$, and there is an offset of two between the time index n and the month index

- $y[n]$ represents the balance in the account at the start of the $n+2^{nd}$ month.

- We have $\rho = 1 + \frac{\frac{6}{12}}{100} = 1.005$

- Since the owner withdraws \$100 per month at the start of month 13 ($n = 11$)

- We may express the input to the system as $x[n] = -100u[n-11]$, we get

$$X(z) = \frac{-100z^{-11}}{1-z^{-1}}$$

- We get

$$Y(z) = \frac{-100z^{-11}}{(1-z^{-1})(1-1.005z^{-1})} + \frac{1.005(10,000)}{1-1.005z^{-1}}$$

- After a partial fraction expansion we get

$$Y(z) = \frac{20,000z^{-11}}{1-z^{-1}} + \frac{20,000z^{-11}}{1-1.005z^{-1}} + \frac{10,050}{1-1.005z^{-1}}$$

- Monthly account balance is obtained by inverse z-transforming $Y(z)$ We get

$$y[n] = 20,000u[n-11] - 20,000(1.005)^{n-11}u[n-11]$$
$$+ 10,050(1.005)^{n}u[n]$$

- The last term $10,050(1.005)^{n}u[n]$ is the natural response with the initial balance
- The account balance
- The natural balance
- The forced response

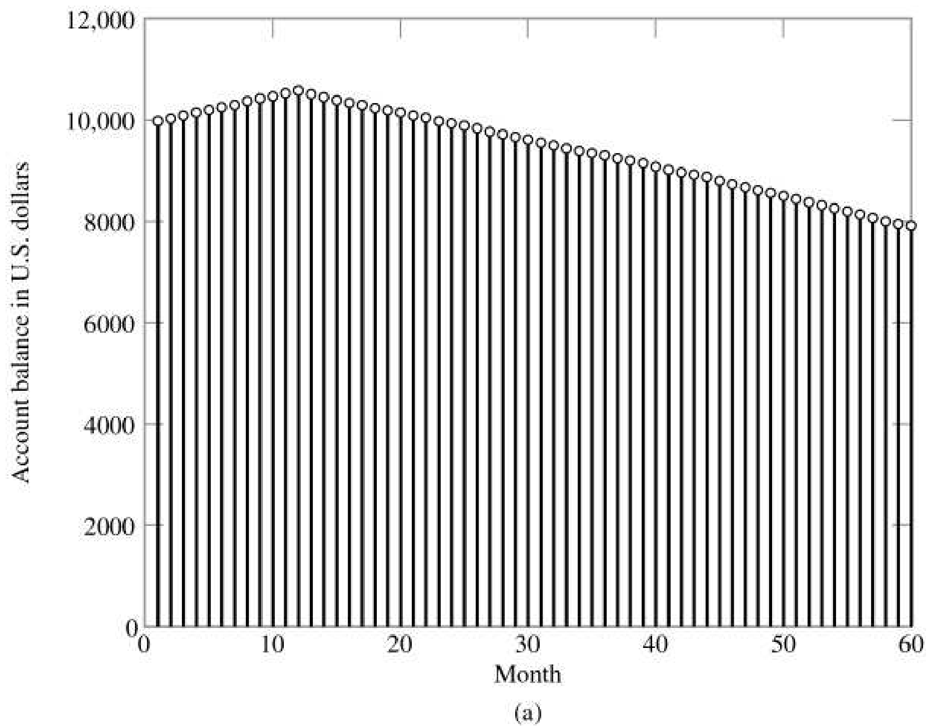


Figure 1.74: Account balance

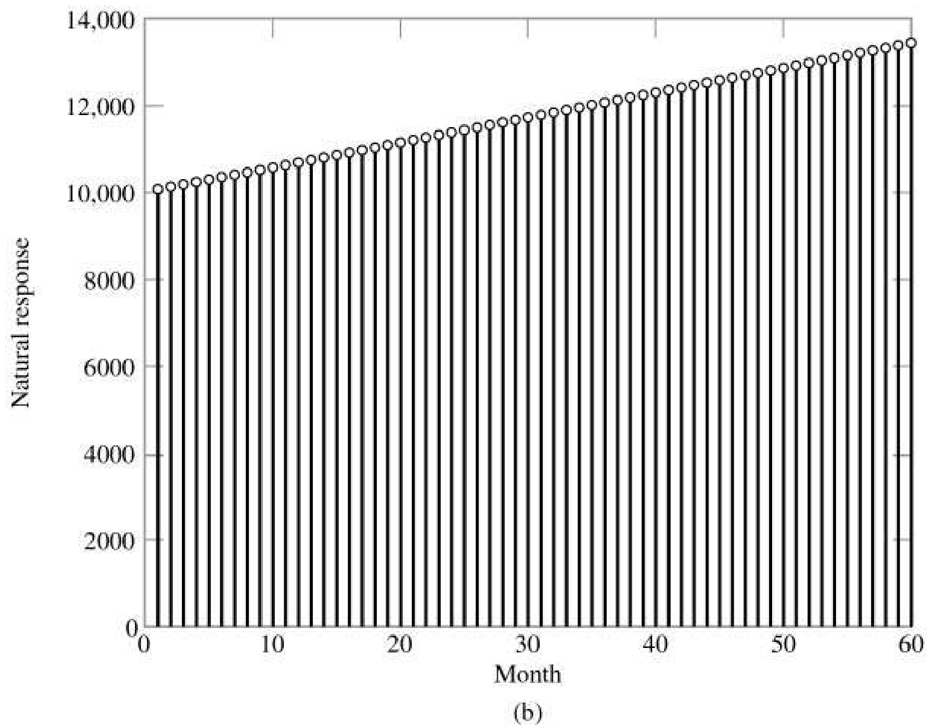


Figure 1.75: The natural response

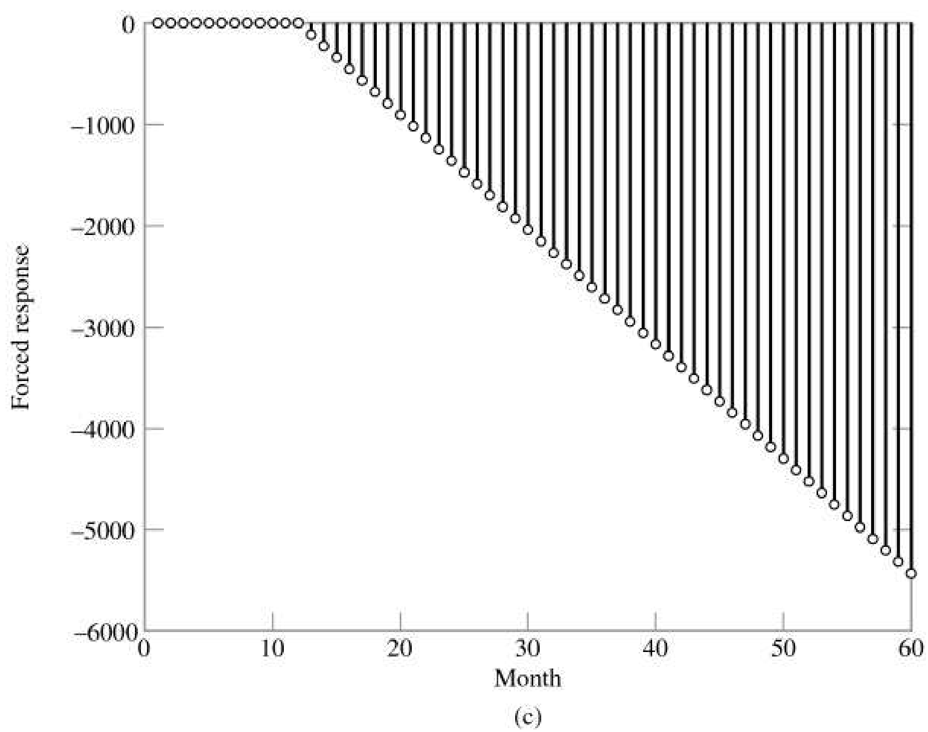


Figure 1.76: The forced response

Example 2

- Determined the forced response $y^{(f)}[n]$, the natural response $y^{(n)}[n]$ and output $y[n]$ of the system described by the difference equation

$$y[n] + 3y[n-1] = x[n] + x[n-1]$$

  if the input is $x[n] = (\frac{1}{2})^n u[n]$ and $y[-1] = 2$ is the initial condition

- Solution: Find the unilateral z-transform and then write in the partial fraction form and find the out $y[n]$, the forced response and the natural response

- The output is

$$y[n] = y^{(f)}[n] + y^{(n)}[n]$$

- The forced response

$$y^{(f)}[n] = \frac{4}{7}(-3)^n u[n] + \frac{3}{7}(\frac{1}{2})^n u[n]$$

- the natural response

$$y^{(n)}[n] = -6(-3)^n u[n]$$

1.10.4 Unsolved examples from [2]

Unsolved Ex. 7.41(a)

- Find the unilateral z-transform of $x[n] = u[n+4]$

$$x[n] = u[n+4] \stackrel{z_u}{\longleftrightarrow} X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} z^{-n}$$

$$X(z) = \frac{1}{1-z^{-1}}$$

Unsolved Ex. 7.41(b)

- Find the unilateral z-transform of $w[n] = x[n-2]$ using (a) and time-shift property

$$w[n] = x[n-2] \stackrel{z_u}{\longleftrightarrow} W(z) = x[-2] + x[-1]z^{-1} + z^{-2}X(z)$$

$$W(z) = 1 + z^{-1} + \frac{z^{-2}}{1-z^{-1}}$$

Unsolved Ex. 7.42(a)

- Use the unilateral z-transform to determine the forced response, the natural response, and the complete response of the system described by the difference equation

$$y[n] - \frac{1}{3}y[n-1] = 2x[n]$$

$$y[-1] = 1, x[n] = (-1)^n u[n]$$

- Solution: take z-transform of $x[n]$ is

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}}$$

- Take z-transform

$$Y(z) - \frac{1}{3}(z^{-1}Y(z)+1) = 2X(z)$$

$$Y(z)(1-\frac{1}{3}z^{-1}) = \frac{1}{3} + 2X(z)$$

- z-transform

$$Y(z) = \frac{1}{3}\frac{1}{1-\frac{1}{3}z^{-1}} + 2\frac{1}{1-\frac{1}{3}z^{-1}X(z)}$$

- We know that

$$Y(z) = Y^{(n)}(z) + Y^{(f)}(z)$$

- The z-transform of natural response is

$$Y^{(n)}(z) = \frac{1}{3}\frac{1}{1-\frac{1}{3}z^{-1}}$$

- The natural response is

$$y^{(n)}[n] = \frac{1}{3}(\frac{1}{3})^{n}u[n]$$

- The z-transform of forced response is

$$Y^{(f)}(z) = 2\frac{1}{1-\frac{1}{3}z^{-1}}X(z) = 2\frac{1}{1-\frac{1}{3}z^{-1}}\frac{1}{1+\frac{1}{2}z^{-1}}$$

- Write the partial fraction expansion and write the forced response and is given by

$$Y^{(f)}(z) = \frac{\frac{6}{5}}{1+\frac{1}{2}z^{-1}} + \frac{\frac{4}{5}}{1-\frac{1}{3}z^{-1}}$$

- The forced response is

$$y^{(f)}[n] = \frac{6}{5}(-\frac{1}{2})^n u[n] + \frac{4}{5}(\frac{1}{3})^n u[n]$$

- Complete response is

$$y[n] = \frac{6}{5}(-\frac{1}{2})^n u[n] + \frac{17}{15}(\frac{1}{3})^n u[n]$$

Unsolved Ex. 7.42(b)

- Use the unilateral z-transform to determine the forced response, the natural response, and the complete response of the system described by the difference equation

$$y[n] - \frac{1}{9}y[n-2] = 2x[n-1]$$

$$y[-1] = 1, y[-2] = 0, \quad x[n] = 2u[n]$$

- Solution: take z-transform of $x[n]$ is

$$X(z) = \frac{2}{1-z^{-1}}$$

- Take z-transform

$$Y(z) - \frac{1}{9}(z^{-2}Y(z) + z^{-1}) = z^{-1}X(z)$$

$$Y(z)(1 - \frac{1}{9}z^{-2}) = \frac{1}{9}z^{-1} + z^{-1}X(z)$$

- z-transform $Y(z)$ is

$$Y(z) = \frac{1}{9}\frac{z^{-1}}{1-\frac{1}{9}z^{-2}} + \frac{z^{-1}X(z)}{1-\frac{1}{9}z^{-2}}$$

- We know that

$$Y(z) = Y^{(n)}(z) + Y^{(f)}(z)$$

- The z-transform of natural response is

$$Y^{(n)}(z) = \frac{1}{9}\frac{z^{-1}}{1-\frac{1}{9}z^{-2}}$$

$$Y^{(n)}(z) = \frac{1}{6}\frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{6}\frac{1}{1+\frac{1}{3}z^{-1}}$$

- The natural response is

$$y^{(n)}[n] = \frac{1}{6}(\frac{1}{3})^n u[n] - \frac{1}{6}(-\frac{1}{3})^n u[n]$$

- The z-transform of forced response is

$$Y^{(f)}(z) = \frac{\frac{9}{4}}{1-z^{-1}} - \frac{\frac{3}{4}}{1+\frac{1}{3}z^{-1}} - \frac{\frac{3}{2}}{1-\frac{1}{3}z^{-1}}$$

- Write the partial fraction expansion and write the forced response and is given by

$$y^{(f)}[n] = \frac{9}{4}u[n] - \frac{3}{4}(-\frac{1}{3})^n u[n] - \frac{3}{2}(\frac{1}{3})^n u[n]$$

Unsolved Ex. 7.42(c)

- Use the unilateral z-transform to determine the forced response, the natural response, and the complete response of the system described by the difference equation

$$y[n]-\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=x[n]+x[n-1]$$

$$y[-1]=1, y[-2]=-1, \quad x[n]=3^{n}u[n]$$

- Solution: take z-transform of $x[n]$ is

$$X(z)=\frac{1}{1-3z^{-1}}$$

- Take z-transform

$$Y(z)-\frac{1}{4}(z^{-1}Y(z)+1)-\frac{1}{8}(z^{-2}Y(z)+z^{-1}-1)$$

$$=X(z)+z^{-1}X(z)$$

- z-transform $Y(z)$ is

$$Y(z)(1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2})=\frac{1}{8}+\frac{1}{8}z^{-1}+(1+z^{-1})X(z)$$

$$Y(z)=\frac{1}{8}\frac{1}{(1+\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}+\frac{(1+z^{-1})X(z)}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$$

- We know that

$$Y(z)=Y^{(n)}(z)+Y^{(f)}(z)$$

- The z-transform of natural response is

$$Y^{(n)}(z) = \frac{1}{4}\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{8}\frac{1}{1+\frac{1}{4}z^{-1}}$$

- The natural response is

$$y^{(n)}[n] = \frac{1}{4}(\frac{1}{2})^n u[n] - \frac{1}{8}(-\frac{1}{4})^n u[n]$$

- The z-transform of forced response is

$$Y^{(f)}(z) = \frac{\frac{96}{65}}{1-3z^{-1}} - \frac{\frac{2}{5}}{1-\frac{1}{2}z^{-1}} - \frac{\frac{1}{13}}{1+\frac{1}{4}z^{-1}}$$

- Write the partial fraction expansion and write the forced response and is given by

$$y^{(f)}[n] = \frac{96}{35}(3)^n u[n] - \frac{2}{5}(-\frac{1}{2})^n u[n] - \frac{1}{13}(-\frac{1}{4})^n u[n]$$

Continuous time

- Rewrite the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

as an integral equation. Let $v^{(0)}(t) = v(t)$ be an arbitrary signal, and set

$$v^{(n)}(t) = \int_{-\infty}^{t} v^{(n-1)}(\tau)d\tau, \qquad n = 1,2,3,\ldots$$

where $v^{(n)}(t)$ is the n-fold integral of $v(t)$ with respect to time

- Rewrite in terms of an initial condition on the integrator as

$$v^{(n)}(t) = \int_0^t v^{(n-1)}(\tau)d\tau + v^{(n)}(0), \quad n = 1,2,3,\ldots$$

- If we assume zero ICs, then differentiation and integration are inverse operations, ie.

$$\frac{d}{dt}v^{(n)}(t) = v^{(n-1)}(t), \quad t > 0 \text{ and } n = 1,2,3,\ldots$$

- Thus, if $N \geq M$ and integrate N times, we get the integral description of the system

$$\sum k = 0^N a_k y^{(N-k)}(t) = \sum k = 0^M b_k x^{(N-k)}(t)$$

- For second order system with $a_0 = 1$, the differential equation can be written as

$$y(t) = -a_1 y^{(1)}(t) - a_0 y^{(2)}(t) + b_2 x(t) + a_1 x^{(1)}(t) + b_0 x^{(2)}(t)$$

Conclusions

- Unilateral z-transform
- Properties of unilateral z-transform
- Solving difference equation with initial condition
- Implementations of continuous time systems

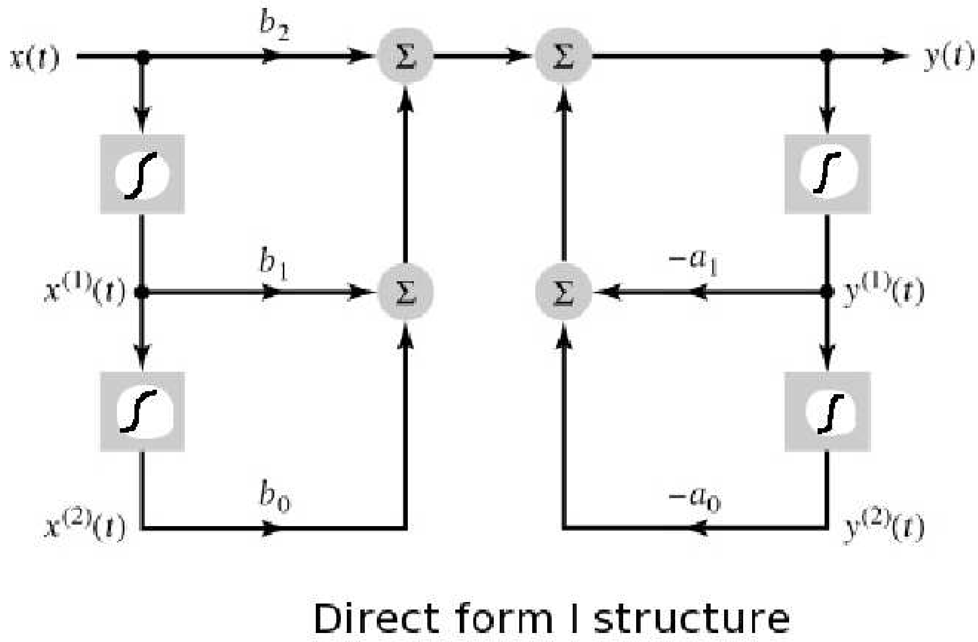


Figure 1.77: Direct form I structure

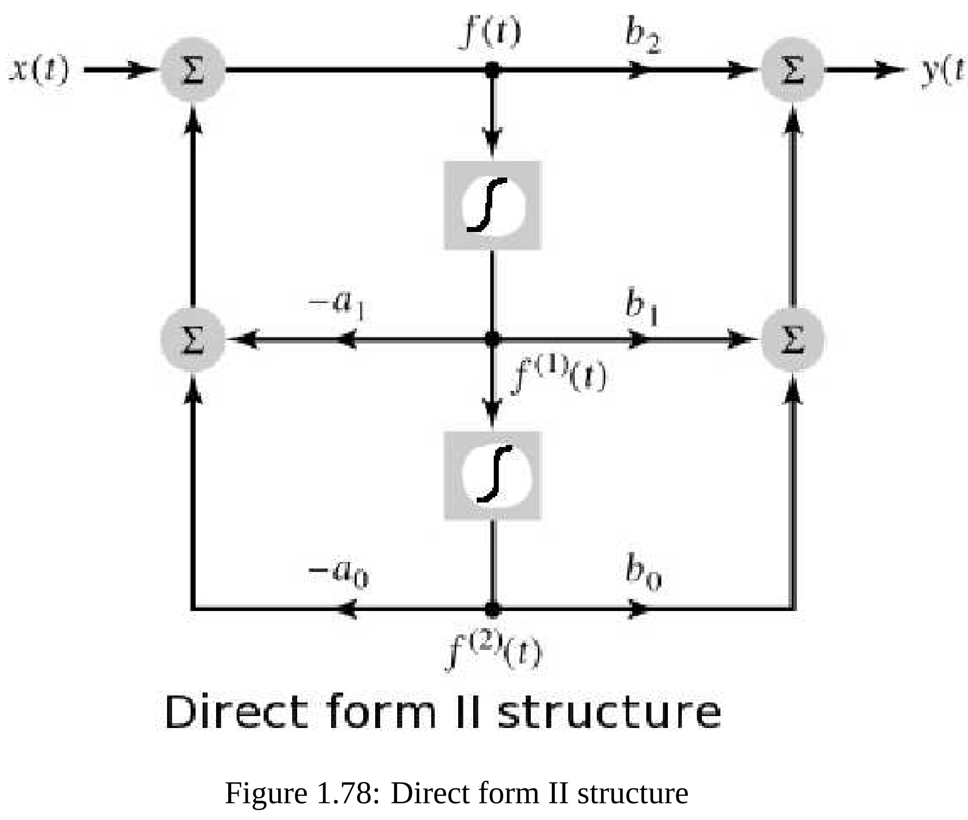


Figure 1.78: Direct form II structure

1.11 Class 11: Discussion of questions from previous question papers

Outline of today's class

- Solution to question papers
- Unsolved examples from the book

1.11.1 Theory part from the question papers

- Signal: power, energy, symmetric, non-symmetric, even, odd, periodic, aperiodic
- Operations on signal
- Signal types: step, impulse, ramp

Systems

- Systems representations: (i) impulse response- characterizes the behavior (ii)Linear constant coefficient differential or difference equation-input output behavior (ii) Block diagram- as an interconnection of three elementary operations
- Systems characteristics: stability, causality, BIBO, time invariance, linearity, memory-less

Convolution

- For an LTI system

- $y(t) = x(t) * h(t)$
- $y[n] = x[n] * h[n]$

Sampling theorem

- Why sampling
- Sampling frequency greater than twice the maximum frequency in the signal
- Aliasing

Fourier series and transform

- Continuous and Periodic: use FS
- Continuous and Non-periodic: use FT
- Discrete and Non-periodic: use DTFT
- Discrete and periodic: use DTFS

1.11.2 Unsolved example from [2]

Unsolved example 1.43

- A signal $x(t) = 3\cos(200t + \frac{\pi}{6})$ is passed through a square law device. Find the DC component and fundamental frequency
- Solution: We know $\cos^2\Theta = \frac{1}{2}(\cos 2\Theta + 1)$

$$y(t) = x^2(t) = (3\cos(200t + \frac{\pi}{6}))^2$$

$$y(t) = 9\cos^2(200t + \frac{\pi}{6})$$

- We get

$$y(t) = 9/2\cos(400t + \frac{\pi}{3} + 1)$$

- DC component:$9/2$, Sinusoidal component:$9/2\cos(400t + \frac{\pi}{3}$ Amplitude:$9/2$, Fundamental frequency: $\frac{200}{\pi}$

Unsolved example 1.46

- Find the total energy of a raised-cosine pulse

$$x(t) = \begin{cases} \frac{1}{2}(\cos(wt) + 1), & -\pi/w \leq t \leq \pi/w \\ 0, & \text{otherwise} \end{cases}$$

- Solution: The energy is

$$E = \int_{-\pi/w}^{\pi/w} \frac{1}{4}(\cos(wt) + 1)^2 dt$$

- The energy is

$$E = \int_0^{\pi/w} \frac{1}{2}(\cos^2(wt) + 2\cos(wt) + 1)dt$$

$$E = \frac{1}{2}\int_0^{\pi/w} (\frac{1}{2}\cos(2wt) + \frac{1}{2} + 2\cos(wt) + 1)dt$$

$$E = \frac{1}{2}\frac{3}{2}\frac{\pi}{w} = 3\pi/4w$$

Unsolved example 1.71

- Given a time varying RC system. Is the system linear?

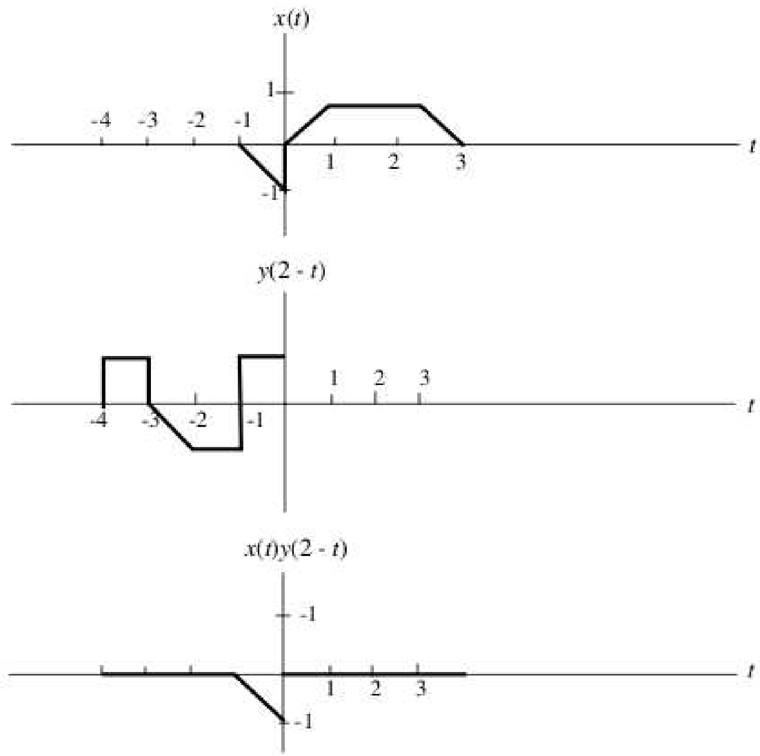


Figure 1.79: Unsolved ex 1.52(e) $x(t)y(2-t)$

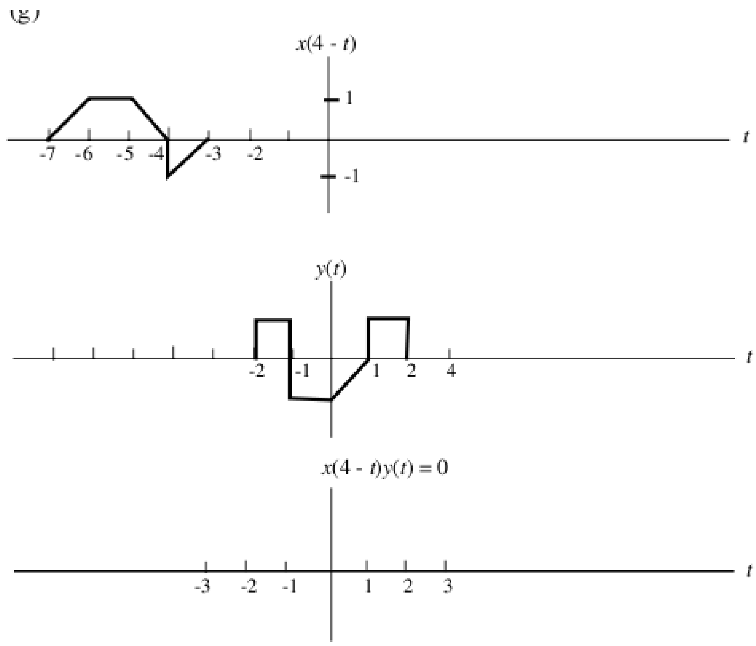


Figure 1.80: Unsolved ex 1.52(g) $x(t)y(2-t)$

- Doubling the input results in doubling the output across capacitor. Hence, the property of homogeneity is satisfied
- The property of superposition is also satisfied.

Unsolved ex. 2.41(b)

- Given RC system, find the effect of ISI for the input sequence ”1110” and ”1000”. Here ”1” is transmitted with the +ve pulse and ”0” with the -ve pulse of duration T sec. Assume $RC = 1/T$, $RC = 5/T$ and $RC = 1/(5T)$. The channel is an ideal channel ($h(t) = \delta(t)$)
- We have

$$x(t) = p(t) + p(t-1) + p(t-2) - p(t-3)$$

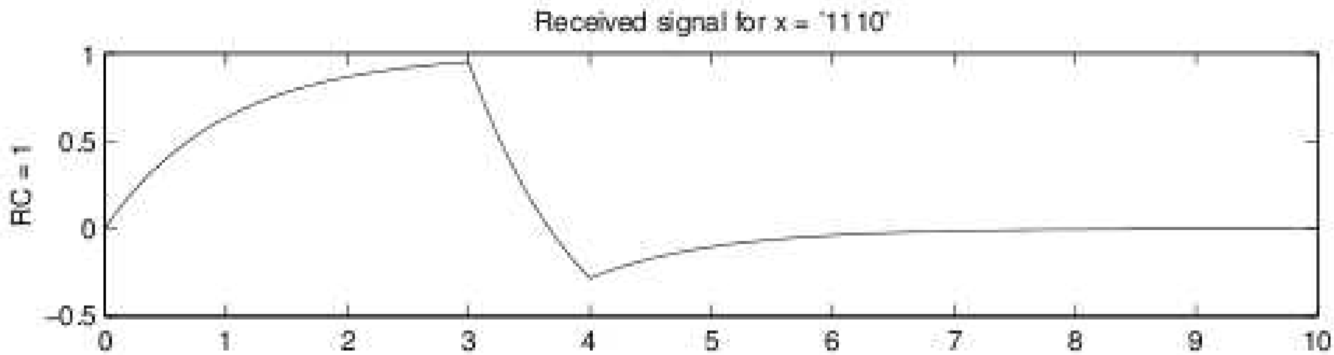


Figure 1.81: For "1110", assume $T = 1$, for $RC = 1$

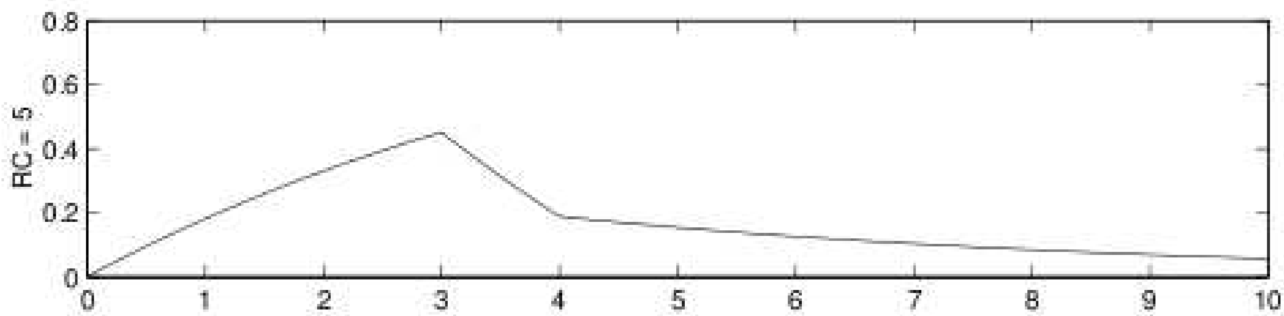


Figure 1.82: For "1110", assume $T = 1$, for $RC = 5$

$$y(t) = y_p(t) + y_p(t-1) + y_p(t-2) + y_p(t-3)$$

- For "1110", assume $T = 1$, for $RC = 1$
- For "1110", assume $T = 1$, for $RC = 5$
- For "1110", assume $T = 1$, for $RC = 1/5$
- For "1000", assume $T = 1$, for $RC = 1$
- For "1000", assume $T = 1$, for $RC = 5$
- For "1000", assume $T = 1$, for $RC = 1/5$

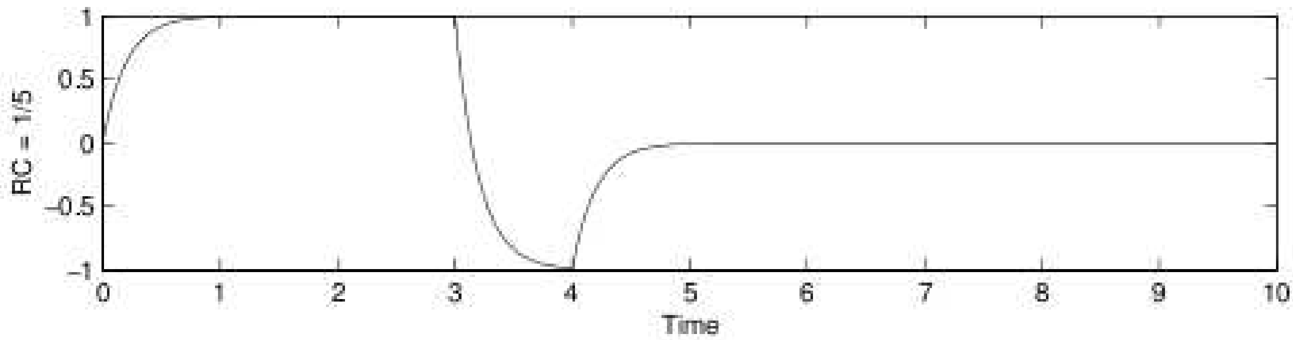


Figure 1.83: For ”1110”, assume $T = 1$, for $RC = 1/5$

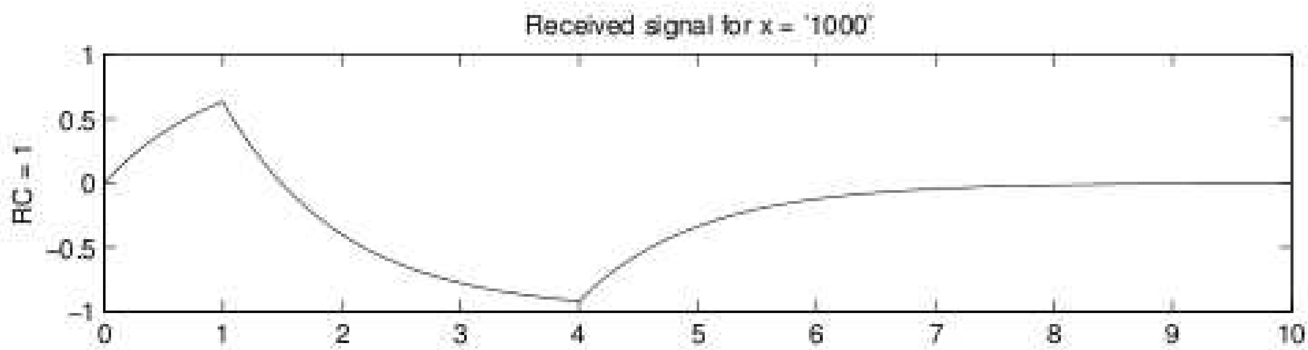


Figure 1.84: For ”1000”, assume $T = 1$, for $RC = 1$

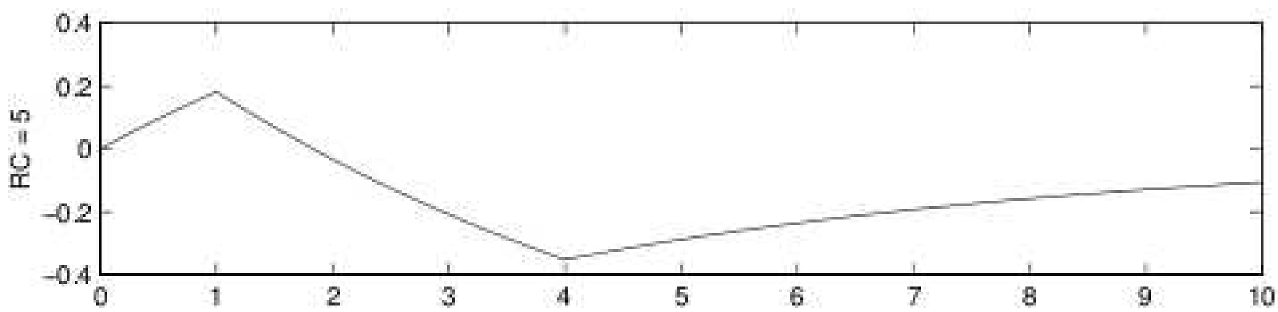


Figure 1.85: For ”1000”, assume $T = 1$, for $RC = 5$

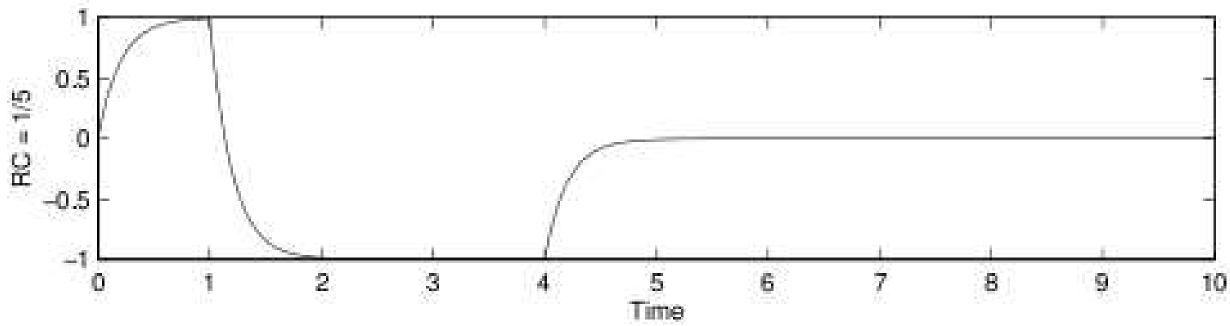


Figure 1.86: For ”1000”, assume $T = 1$, for $RC = 1/5$

Unsolved ex. 2.49

- For a given impulse response, determine whether the corresponding system is
  - memory-less: if and only if $h(t) = c\delta(t)$
  - causal: if and only if $h(t) = 0$ for $t < 0$
  - stable: Stable if and only if $h(t)$ is bounded.

Unsolved ex. 2.49(b)

- $h(t) = e^{-2t}u(t-1)$
- (i) has memory
- (ii) causal
- (iii) stable

Unsolved ex. 2.49(d)

- $h(t) = 3\delta(t)$

- (i) memory-less
- (ii) causal
- (iii) stable

Unsolved ex. 2.49(f)

- $h[n] = (-1)^n u[-n]$
- (i) has memory
- (ii) not causal
- (iii) not stable

Unsolved ex. 2.51

- Suppose the multipath propagation model is generalized to a k-step delay between the direct and reflected paths as shown by the input-output equation $y[n] = x[n] + ax[n-k]$ Find the impulse response of the inverse system.
- Solution:

$$h^{inv}[n] + ah^{inv}[n-k] = \delta[n]$$

$$h^{inv}[0] + ah^{inv}[-k] = 1]$$

- For the causal system

$$h^{inv}[0] = 1$$

- Which means $h^{inv}[n]$ is nonzero only for positive multiples of k

$$h^{inv}[n] = -ah^{inv}[n-k]$$

$$h^{inv}[n] = \sum_{p=0}^{\infty} (-a)^p \delta[n - pk]]$$

Unsolved ex. 2.58(a)

Identify the natural response for the system: $\frac{d}{dt}y(t) + 10y(t) = 2x(t)$, $y(0^-) = 1$, $x(t) = u(t)$

- $r + 10 = 0$ and $r = -10$
  $y_n(t) = c_1 e^{-10t}$
  $y(0^-) = 1 = c_1$
  $y_n(t) = e^{-10t}$

Unsolved ex. 2.58(c)

Identify the natural response for the system: $\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = 2x(t)$, $y(0^-) = -1$, $\frac{d}{dt}y(t)|_{t=0^-} = 1, x(t) = e^{-t}u(t)$

- $y_n(t) = c_1 e^{-4t}u(t) + c_2 e^{-2t}u(t)$
  $y(0^-) = -1 = c_1 + c_2$ and $\frac{d}{dt}y(t)|_{t=0^-} = 1 = -4c_1 - 2c_2$
  and $c_1 = \frac{1}{2}, c_2 = -\frac{3}{2}$
  $y_n(t) = \frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t}$

Unsolved ex. 2.59

- Find the output of the system described by the difference equation with input and initial conditions $y[n] - \frac{1}{2}y[n-1] = 2x[n], y[-1] = 3, x[n] = (-\frac{1}{2})^n u[n]$

- Solution: Natural response $n \geq 0$

$$r - \frac{1}{2} = 0 \quad i\text{and} \quad y^{(n)}[n] = c(\frac{1}{2})^n$$

- Particular solution

$$y^{(p)}[n] = k(-\frac{1}{2})^n u[n]$$

$$k(-\frac{1}{2})^n - \frac{1}{2}k(-\frac{1}{2})^{n-1} = (-\frac{1}{2})^n$$

- We get $k = 1$

$$y^{(p)}[n] = (-\frac{1}{2})^n u[n]$$

- Translate initial conditions

$$\frac{7}{2} = 1 + c$$

$$c = \frac{5}{2}$$

- We get $k = 1$

$$y[n] = (-\frac{1}{2})^n u[n] + \frac{5}{2}(\frac{1}{2})^n u[n]$$

Unsolved example 2.65(a)

- Find the difference equation for the system

- $f[n] = -2y[n] + x[n], y[n] = f[n-1] + 2f[n]$ $y[n] = -2y[n-1] + x[n-1] - 4y[n] + 2x[n]$ $5y[n] + 2y[n-1] = x[n-1] + 2x[n]$

Unsolved example 2.65(b)

- Find the difference equation for the system

- $f[n] = y[n] + x[n-1], \qquad y[n] = f[n-1] = y[n-1] + x[n-2]$

Unsolved example 2.65(c)

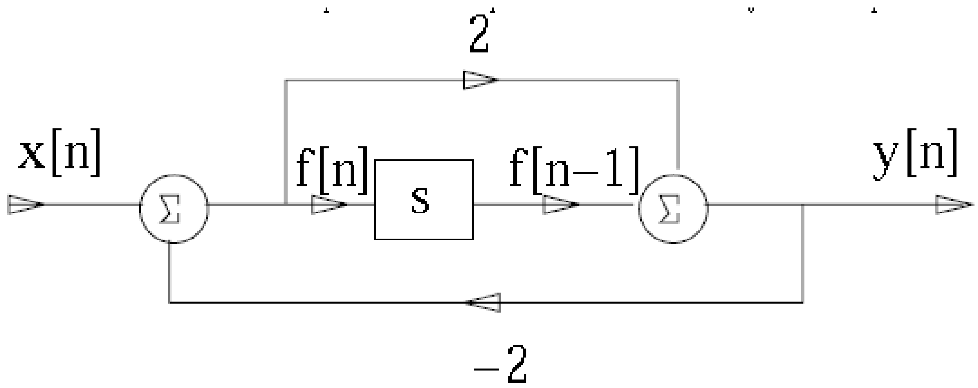


Figure 1.87: Unsolved example 2.65(a)

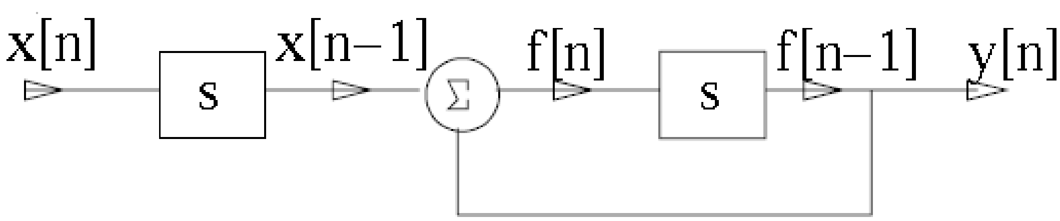


Figure 1.88: Unsolved example 2.65(b)

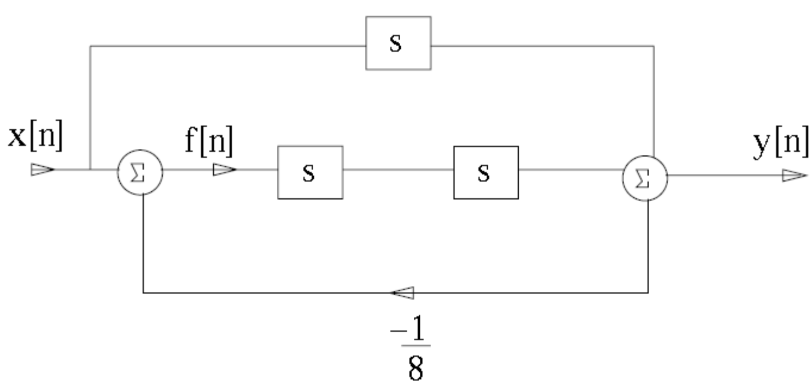


Figure 1.89: Unsolved example 2.65(c)

- Find the difference equation for the system
- $f[n] = x[n] - (1/8)y[n], y[n] = x[n-1] + f[n-2]$ $y[n] + (1/8)y[n-2] = x[n-1] + x[n-2]$

Conclusions

- Unit I: Introduction
- Units II and III: Time domain Analysis
- Units IV, V, VI: Fourier representation
- Units VII, VIII: Z-domain Analysis

Bibliography

[1] R. Ranjan H. P Hsu. *Signals and Systems*. Schams outlines, TMH, 2006.

[2] Simon Haykin and Barry Van Veen. *Signals and Systems*. John Wiley and Sons, Inc, II Ed. 2001, Reprint 2002.

[3] B. P. Lathi. *Linear Systems and Signals*. Oxford University Press, 2005.

[4] Alan V. Oppenheim, Alan S. Willsky, and A. Hamid Nawab. *Signals and Systems*. Pearson Education Asia / PHI, 2nd edition, 1997, Indian Reprint 2002.

[5] Ganesh Rao and Satish Tunga. *Signals and Systems*. Sanguine Technical Publishers, 2004.